UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2007

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

STRUCTURE OF MATTER, VIBRATIONS & WAVES and QUANTUM PHYSICS

For First-Year Physics Students

Wednesday 6th June 2007: 10.00 to 12.00

Answer ALL parts of Section A, ONE question from Section B, ONE question from Section C and ONE question from Section D.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the SIX answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in SIX answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) Use the theorem of equipartition of energy to find the internal energy U of one mole of an ideal gas at temperature T. Hence obtain an expression for the molar specific heat at constant volume C_v of an ideal gas.

[2 marks]

(ii) State the relation between pressure and volume for an adiabatic process on an ideal gas. An ideal gas at 20°C and one atmosphere of pressure is compressed adiabatically to a pressure of ten atmospheres. What fraction of the initial volume does the gas now occupy? Calculate the final temperature.

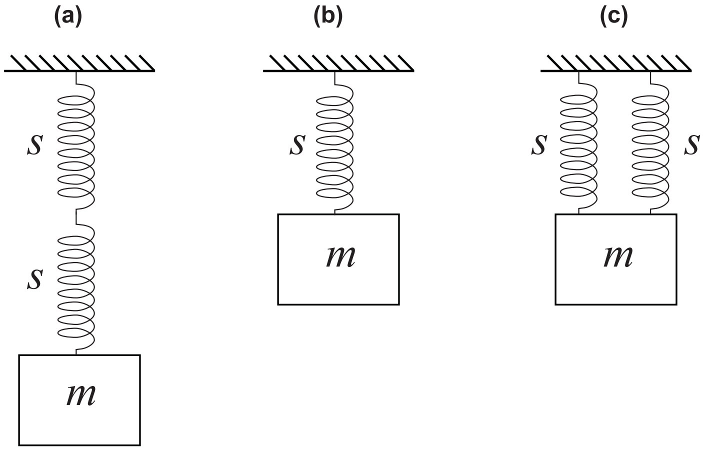
[3 marks]

(iii) State Archimedes principle. A cube of ice with sides of 1 m in length and density $920\,\text{kg/m}^3$ is submerged in water of density $1000\,\text{kg/m}^3$. If the cube is upright (i.e. its sides are all vertical), calculate the height above the water surface of the top of the cube. (You may assume that the acceleration due to gravity is $g = 10\,\text{N/kg}$.)

[3 marks]

[TOTAL 8 marks]

2. (i) What are the periods of the three simple harmonic oscillators (a), (b), (c) below? Each spring has a spring constant of s and m is the mass.



[2 marks]

(ii) According to classical electromagnetic theory, an accelerating electron radiates power $P = Ka^2$ where K is a constant and a the acceleration. If the displacement of the electron is given by $x(t) = A\cos(\omega t)$, show that the energy lost in one period T of oscillation is given by,

$$E_{lost} = K\pi A^2\omega^3 \ .$$

You may use $\int_0^T \cos^2(\omega t)dt = T/2$.

[3 marks]

(iii) The equation of a transverse travelling wave on a stretched string is given by,

$$\psi(z, t) = 4.2 \times 10^{-3} \cos[\pi(100t - 100z)] \ ,$$

where ψ and z are in metres and t is in seconds. Determine the amplitude A, the frequency f, the wavelength λ and the phase velocity v_p of the wave. What is the maximum transverse speed of any small segment of the string?

[3 marks]

[TOTAL 8 marks]

3. (i) The lifetimes of the 2s and 2p excited states of the hydrogen atom are 0.1 s and 3×10^{-9} s, respectively. Estimate the corresponding energy uncertainties.

[3 marks]

(ii) A relativistic electron beam is to be used to investigate the internal structure of a nucleus of radius 2×10^{-15} m. Estimate the minimum electron momentum and energy required.

[3 marks]

(iii) Normalise the following wavefunction:

$$\psi(x) = \begin{cases} 0 & x < -a, \\ x^2 - a^2 & -a \leq x \leq a, \\ 0 & x > a. \end{cases}$$

[3 marks]

[TOTAL 9 marks]

SECTION B

4. (i) The interaction between a pair of atoms can be approximated by the Lennard-Jones (6-12) potential:

$$U(r) = \epsilon \left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6} \right] ,$$

where r is the separation of the centre of the atoms. Show that the equilibrium separation distance is r_0. Calculate the potential energy for this separation of the atoms. Sketch the form of the potential energy, showing regions that correspond to *attractive* and *repulsive* forces between the atoms.

[8 marks]

(ii) The density of solid argon at its melting point (84 K) is 1700 kg/m^3. Given that the atomic mass of Ar atoms is 39.9, calculate the volume occupied by each atom. Assuming that each atom occupies an equivalent cubic volume, calculate the length of the cube, and hence the interatomic spacing r_0 of solid argon.

[4 marks]

(iii) Assuming that argon is an ideal gas at 87 K, find the number density of argon gas just above its boiling point at a pressure of 1 atmosphere (=1.01×10^5 Pa). Hence find the interatomic spacing of argon atoms as a gas at this temperature. What is the ratio of the spacing in the gaseous and solid phases?

[5 marks]

(iv) The interatomic potential of Ar atoms is well described by the Lennard-Jones potential. Calculate the potential energy due to a pair of argon atoms at their mean interatomic spacing as a gas (at 87 K) compared to the potential at their equilibrium separation. Discuss *briefly* whether the assumption that argon is an ideal gas is justified.

[4 marks]

(v) Argon melts and vapourises at similar temperatures (84 K and 87 K). Hence it is possible to calculate the binding energy of an atom in the solid as the sum of the latent heat of fusion and latent heat of vaporisation. Use this, and the fact that each argon atom has 12 nearest neighbours to calculate the binding energy ϵ. For argon $L_{fusion} = 1.18$ kJ/mol, $L_{vap} = 6.43$ kJ/mol, and you may assume that the change in kinetic energy can be neglected.

[4 marks]

[TOTAL 25 marks]

5. (i) The Maxwell-Boltzmann distribution for the speed of particles in a gas at a temperature T is given by,

$$f(v)\,dv = 4\pi\left(\frac{m}{2\pi k_b T}\right)^{3/2} v^2 e^{-mv^2/2k_bT}\,dv\ .$$

Sketch the shape of this curve. Indicate on the graph the most probable speed v_{mp}, and the value of the distribution function at this speed. [6 marks]

(ii) On the same sketch, indicate what would happen to the distribution if the temperature were doubled to $2T$. What happens to v_{mp} in this case, and the value of $f(v_{mp})$? [4 marks]

(iii) Find the mean velocity $\bar{v}$ of the distribution for a temperature T. You may use the standard integral,

$$\int_0^{+\infty} x^3 e^{-\alpha x^2} dx = \frac{1}{2\alpha^2}\ .$$

[6 marks]

(iv) Find the root mean square velocity v_{rms} for temperature T. You may use the standard integral,

$$\int_0^{+\infty} x^4 e^{-\alpha x^2} dx = \frac{3}{8}\left(\frac{\pi}{\alpha^5}\right)^{1/2}\ .$$

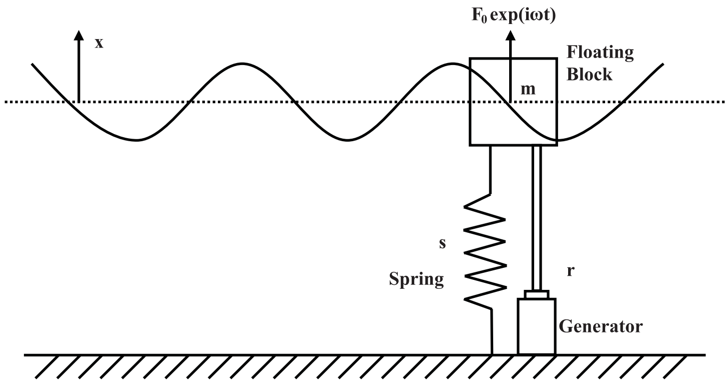
[6 marks]

(v) Compare the values v_{mp}, $\bar{v}$ and v_{rms} putting them in order of magnitude. [3 marks]

[TOTAL 25 marks]

SECTION C

6. You are asked to help with the design of a machine to extract energy from ocean waves arriving at the coast. A floating block of total mass m is tethered to the seabed by a spring of spring constant s. A rigid rod connects the block to an electrical generator on the seabed with damping constant r. Surface water waves exert a vertical force on the block $F_0 \exp(i\omega t)$, where F_0 is real.



(i) The equation of motion for the vertical displacement $\tilde{x}$ of the floating block is given by,

$$m\frac{d^2\tilde{x}}{dt^2} + r\frac{d\tilde{x}}{dt} + s\tilde{x} = F_0 \exp(i\omega t) \ .$$

Briefly explain the physical significance of each term on the left hand side of the equation.

[3 marks]

(ii) Show that the equation of motion has a solution $\tilde{x}(t) = A\exp[i(\omega t + \varphi)]$ where A and φ are both real and satisfy

$$\left(\omega_0^2 - \omega^2\right) A + \frac{i\omega r}{m} A = \frac{F_0}{m}\exp(-i\varphi) \ ,$$

where $\omega_0^2 = s/m$. What is ω_0?

[5 marks]

(iii) By equating real and imaginary parts, show that

$$A(\omega) = \frac{F_0/m}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \left(\frac{\omega r}{m}\right)^2}}$$

and

$$\tan[\varphi(\omega)] = \frac{-\omega r}{m\left(\omega_0^2 - \omega^2\right)} \ .$$

[5 marks]

Question 6 continued overleaf

6. (continued)

(iv) Show that the resonant frequency ω_{res} where maximum displacement of the block occurs is given by,

$$\omega_{res} = \sqrt{\omega_0^2 - \frac{r^2}{2m^2}} \, .$$

[3 marks]

(v) Sketch $A(\omega)$ as a function of ω. How in general terms would you ensure that the machine works efficiently for waves of a broad range of frequencies? [3 marks]

(vi) The machine is specified to operate with a maximum amplitude of 0.5 m at very low frequencies (well below resonance) and to have a resonant frequency of 1.41 rad/s. If the mass of the block is $m = 1000$ kg and the maximum force exerted by the waves on the block is 2000 N what values of s and r are required? What is the maximum displacement at resonance?

[6 marks]

[TOTAL 25 marks]

7. A horn is made from a pipe of length L open at both ends. Standing sound waves are formed in the pipe by blowing air across a small hole in the side of the tube.

(i) A sound wave travelling in the positive z direction in the pipe is given by,

$$\psi_1(z,t) = A\cos(\omega t - kz)\,.$$

When the wave encounters the open end at $z = L$, it is reflected without change in amplitude (no inversion) to give a wave $\psi_2(z,t)$ travelling in the negative z direction. Write down an expression for $\psi_2(z,t)$. Hence show that a standing sound wave is produced in the pipe of the form,

$$\psi(z,t) = 2A\cos(kz)\cos(\omega t)\,.$$

[5 marks]

(ii) An open pipe has antinodes (pressure maxima) at the open ends. Hence show that the pipe will support standing waves of wavelength $\lambda_n = 2L/n$ where n is a positive integer. Sketch, on the same axes, the $n = 1$ mode between $z = 0$ and $z = L$, at times $t_1 = 0$, $t_2 = \pi/3\omega$ and $t_3 = \pi/2\omega$.

[5 marks]

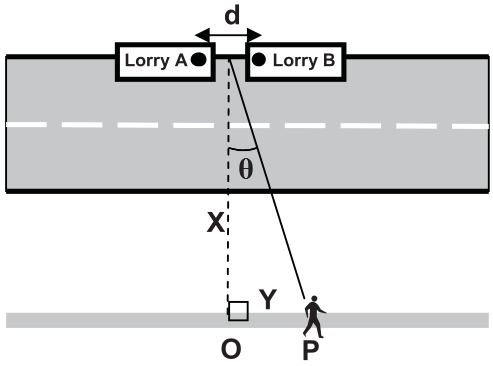
(iii) What length of pipe gives a lowest frequency standing wave of $f_1 = 500\,\text{Hz}$? Take the speed of sound to be $340\,\text{m/s}$.

[3 marks]

Question 7 continued overleaf

7. (continued)

Two lorries fitted with identical horns ($f_1 = 500\,\text{Hz}$) are parked at the side of a straight section of road. A pedestrian is walking along a path parallel to the road $X = 20\,\text{m}$ away.



(iv) The lorries sound their horns together (assume they are in phase and have the same amplitude). At the midpoint between the lorries, O, the pedestrian hears a maximum in the sound intensity. She then walks a distance $Y = 4\,\text{m}$ to point P to reach the next maximum. What is the separation, d, of the horns?

[6 marks]

(v) Lorry B then drives away sounding its horn. The Doppler shifted frequency received by the stationary lorry driver is,

$$f' = \left(\frac{v_p}{v_p + v}\right) f_1 \,,$$

where v_p is the phase velocity and v is the speed of the lorry.

The driver of the stationary lorry sounds his horn at the same time as the moving lorry and notices a beat of frequency f_{beat}. Show that f_{beat} is given by,

$$|f_{beat}| = \left(\frac{v}{v_p + v}\right) f_1 \,.$$

[4 marks]

(vi) How fast was the lorry B moving if $f_{beat} = 40.5\,\text{Hz}$?

[2 marks]

[TOTAL 25 marks]

SECTION D

8. (i) Write down the Planck and de Broglie equations. [4 marks]

(ii) Write down the dispersion relation of a photon in vacuum. Hence derive the relationship between the energy E and momentum p of a photon. [4 marks]

(iii) When an electron and a positron (both of mass $511\,\mathrm{keV}/c^2$) annihilate, two photons are produced. Why not one? [2 marks]

(iv) Assuming that the electron and positron are at rest when they annihilate, find the energies, momenta, wavelengths and frequencies of the two photons. Why do they have the same energy? [6 marks]

(v) One measure of the efficiency of a spacecraft engine is the *specific impulse*, defined as the momentum gained by the spacecraft per unit weight (on earth) of fuel used. The specific impulse is measured in seconds.

The engine of the space shuttle has an exhaust velocity of $4500\,\mathrm{ms}^{-1}$. Show that its specific impulse is approximately $460\,\mathrm{s}$. [4 marks]

(vi) By evaluating the momentum of the photons emitted per unit weight of fuel annihilated, estimate the maximum specific impulse possible for a spacecraft powered by electron-positron annihilation. You may assume for the purposes of this question that the technical problems of storing positrons and making mirrors for gamma ray photons have been solved. [5 marks]

[TOTAL 25 marks]

9. (i) The bond in a nitrogen (N_2) molecule acts like a spring with spring constant $s = 2240\,\mathrm{Nm}^{-1}$. Given that the atomic mass of nitrogen is 14, evaluate the reduced mass μ and the angular frequency ω of the stretching vibration of the molecule.

[5 marks]

(ii) Write down an expression for the energy levels of a quantum mechanical simple harmonic oscillator with angular frequency ω.

[3 marks]

(iii) Estimate the temperature at which the vibrational degrees of freedom of nitrogen begin to contribute to the specific heat capacity of air. Why is the vibrational contribution suppressed below this temperature?

[6 marks]

In a simplified two-dimensional model, the rotational energy levels of an N_2 molecule are the energy eigenvalues of the Schrödinger equation

$$-\frac{\hbar^2}{2\mu a^2}\frac{d^2\psi(\theta)}{d\theta^2} = E\psi(\theta)\ ,$$

where θ is the angle of rotation measured in radians and $a = 1.09\,\text{Å}$ is the distance between the two nitrogen atoms. The rotation is about an axis perpendicular to the bond and μ is the reduced mass as obtained in part (i).

(iv) Show that the trial solution $\psi(\theta) = e^{i\ell\theta}$ satisfies this equation and express the eigenvalue E in terms of ℓ and other constants. Given the boundary condition $\psi(\theta + 2\pi) = \psi(\theta)$, find the allowed values of ℓ.

[7 marks]

(i) Estimate the temperature at which the rotational degrees of freedom of nitrogen begin to contribute to the specific heat capacity of air.

[4 marks]

[TOTAL 25 marks]

End