UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2004

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

STRUCTURE OF MATTER, VIBRATION & WAVES and QUANTUM PHYSICS

For First-Year Physics Students

Wednesday 9th June 2004: 14.00 to 17.00

Answer ALL parts of Section A, ONE question from Section B, ONE question from Section C and ONE question from Section D.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the SIX answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE *ONE* ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in SIX answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

Values of Constants

Boltzmann's constant	k_B	$=$	$1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Acceleration due to gravity	g	$=$	$9.81\,\mathrm{m\,s^{-2}}$
Absolute zero of temperature: $0\,\mathrm{K}$		$=$	$-273^\circ\,\mathrm{C}$

SECTION A (Compulsory)

1. (i) Consider two blocks of iron, one of mass 10 kg and initial temperature of 100° C, the second of mass 5 kg and initial temperature 10° C. They are placed in thermal contact, but thermally insulated from their surroundings, and allowed to reach equilibrium. Calculate their final temperature.

[3 marks]

(ii) An ideal gas of N molecules, at temperature T, undergoes a quasi-static, isothermal volume change from volume V_0 to V_1. Show that the work done on the gas is

$$W = -Nk_BT \ln(V_1/V_0) \ .$$

By using the first law of thermodynamics, write down an expression for the heat into the gas during this process, in terms of T, V_0 and V_1.

[3 marks]

(iii) The Maxwell-Boltzmann speed distribution function for ideal gas molecules of mass m at temperature T is

$$f(v) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-mv^2/2k_BT} \ .$$

Show that the most probable speed of a particle in such a distribution is given by

$$v_{mp} = \left(\frac{2k_BT}{m}\right)^{1/2} \ .$$

[3 marks]

[TOTAL 9 marks]

2. (i) A driven mechanical simple harmonic oscillator of mass m obeys the Equation of Motion:

$$m\frac{d^2x}{dt^2} = -r\frac{dx}{dt} - 100x + 4\cos(\omega t)$$

where all values are in SI units. In SI units what is the restoring force constant s of the oscillator and the amplitude F_0 of the driving force? Sketch the variation of the maximum displacement of the oscillator x_{max} with driving force angular frequency ω for large, medium and small r. If $r = 0$ what is the displacement resonant angular frequency ω_r of the oscillator? [3 marks]

(ii) Show that:

$$\psi(x, t) = A\exp(j(\omega t - kx + \phi))$$

is a solution of the wave equation:

$$\frac{d^2\psi}{dx^2} = \frac{1}{v^2}\frac{d^2\psi}{dt^2}$$

If the angular frequency and wavevector are $2.5 \times 10^{16}\,\mathrm{rad\,s^{-1}}$ and $1 \times 10^{8}\,\mathrm{rad\,m^{-1}}$, what is the wave velocity? If this is an electromagnetic wave, what is the refractive index n of the medium in which it is travelling? [3 marks]

(iii) A stone is dropped into a pond creating a wave pulse containing wavelengths from 0.01 m to 0.09 m and peaked at 0.05 m. The waves are deep-water surface gravity waves whose phase velocity v varies as:

$$v = \sqrt{g/k}$$

where $g = 10\,\mathrm{m\,s^{-2}}$ and k is the wavevector of the waves. What is the phase velocity of the shortest and longest wavelengths in the wave pulse? What is the value of the group velocity v_g of the pulse? [3 marks]

[TOTAL 9 marks]

3. (i) On a dark night, most people can see a 100 W light bulb from at least 1 km away. Given that a 100 W light bulb emits about 5 W of visible light, and assuming that the wavelength is 500 nm, calculate the number of photons per second entering each eye (pupil diameter 0.7 cm) of an observer 1 km from the bulb.

What is the average distance between photons en route from the bulb to the eye?

[3 marks]

(ii) An interference experiment produces a travelling wave of the form

$$\psi(x,t) = \sqrt{3}\cos(kx - \omega t) + \sin(kx - \omega t) \; .$$

Find the complex number A for which

$$\psi(x,t) = \mathrm{Re}\left[A e^{i(kx-\omega t)}\right] \; .$$

Hence find the real constants a and ϕ in the expression $\psi(x,t) = a\cos(kx - \omega t + \phi)$.

[3 marks]

(iii) Write down an expression for the zero-point energy of a quantum mechanical simple harmonic oscillator of angular frequency ω.

The typical vibrational frequency of an atom in a piece of aluminium metal is about 8×10^{12} Hz, and each atom can vibrate in three different directions. Given that the atomic weight of aluminium is 27, estimate the total zero-point energy of 1 kg of aluminium.

[3 marks]

[TOTAL 9 marks]

SECTION B

4. (i) State the theorem of equipartition of energy, relating the average energy of a particle to its number of degrees of freedom.

Use equipartition to write down an expression for the internal energy of an ideal gas containing N particles with n_d degrees of freedom at temperature T. [4 marks]

(ii) State the differential form of the first law of thermodynamics, and use it to show that in a quasi-static, adiabatic process in an ideal gas

$$\frac{dT}{T} = -\frac{2}{n_d}\frac{dV}{V} .$$

[5 marks]

(iii) By integrating the expression obtained in the previous part show that in such a process $PV^{\gamma} =$ constant, where $\gamma = (n_d + 2)/n_d$. [5 marks]

(iv) A monatomic ideal gas initially has a volume of 3 m^3, a temperature of 300 K and is at a pressure of 10^5 Pa. It is compressed adiabatically and quasi-statically to a volume of 2 m^3. Calculate its final pressure and temperature. [4 marks]

[TOTAL 18 marks]

5. (i) A simplified model of the atmosphere treats it as an isothermal ideal gas (temperature T) of infinite height with a single type of molecule, of mass m. The curvature of the Earth's surface is also neglected. By considering a slab of atmosphere of thickness dz at height z, show that in this model the number density of the molecules, n, satisfies the following differential equation:

$$\frac{dn}{dz} = -\frac{mg}{k_B T} n$$

where z is the height above the Earth's surface.

[6 marks]

(ii) The probability of any given molecule being between heights z and $z + dz$ can be written as $p(z)dz$, where $p(z)$ is proportional to $n(z)$. Solve the equation found in part (i) and use the fact that the molecule must be at some height between zero and infinity to show that

$$p(z) = \frac{1}{\lambda} e^{-z/\lambda} \, ,$$

where $\lambda = k_B T / mg$.

[6 marks]

(iii) Given the following standard integral (which you may assume without proof):

$$\int_0^\infty x e^{-\alpha x} \, dx = \frac{1}{\alpha^2}$$

show that the average height of a molecule in the atmosphere is λ.

Calculate the value of λ for $T = 20°$ C and $m = 4.82 \times 10^{-26}$ kg (the average mass per molecule in air).

[6 marks]

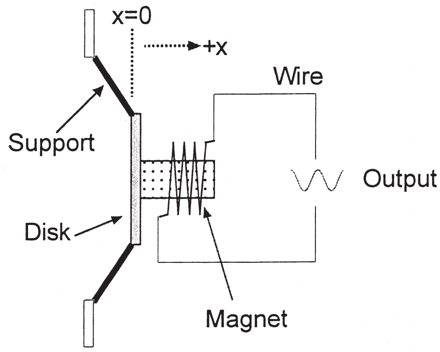
[TOTAL 18 marks]

SECTION C

6. A microphone is constructed consisting of a disk of material attached to a magnet. The disk and magnet have total mass m. A cone-shaped support of flexible material allows the disk to move backwards and forwards in the x-direction. The support produces a restoring force F on the disk to any displacement from equilibrium at $x = 0$ of:

$$F = -sx$$

where s is the spring constant. The microphone output is measured by a current generated by the passage of the magnet through a coil of wire. Assume that the energy lost from the system in generating the current is negligible.



Let $m = 0.02$ kg and $s = 790,000$ N/m

(i) *Initially* assume that there are *no* resistive forces.

(a) What is the angular frequency ω_0 of oscillation of the microphone? What is the frequency f_0 of oscillation? [2 marks]

(b) How does the potential energy PE vary with x when the disk is displaced from equilibrium? [2 marks]

(c) If the disk were to oscillate with an amplitude of 10^{-3} m, what would be the total energy TE stored in the oscillation? [2 marks]

(ii) *Now* assume that the support produces a *resistive force* equal to $-r$ multiplied by the disk's velocity in the x-direction. Let $r = 100\,\text{Ns/m}$.

(a) By solving the Equation of Motion, the displacement of the disk in the *lightly damped regime* has a solution of the form:

$$x(t) = A \exp\left(-\frac{r}{2m}t\right) \cos(\omega' t + \phi).$$

What is the value of ω'? What value of r would be needed if the system was *critically damped*? [3 marks]

(b) A short sound wave impulse is sent to the microphone causing it to be displaced from its equilibrium position with an initial velocity $v_o = 1$ m/s. What are the values of the amplitude A and phase angle ϕ? [2 marks]

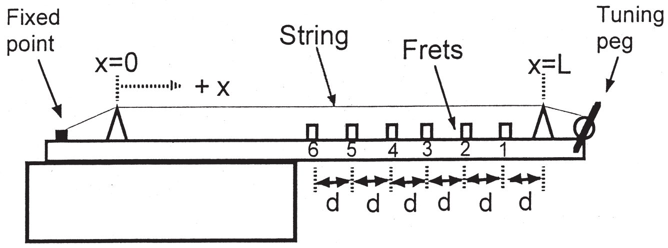
(c) How does the total energy TE of the oscillation vary with time? [2 marks]

(d) Find the quality factor Q of the microphone. [2 marks]

(iii) Over what range of frequencies can the microphone be usefully used? [3 marks]

[TOTAL 18 marks]

7. A guitar string is fixed at one end. The other end it is attached to a tuning peg which, when rotated, allows the tension T^* in the string to be adjusted. The part of the string which vibrates is between two ridges at $x = 0$ and $x = L$. Other lower ridges, called frets, on the guitar neck allow the musician to adjust the length of the part of the string which vibrates by pushing the string down on to the fret with a finger. Let the frets numbered from $m = 1$ to 6 be equally spaced along the guitar neck at intervals of d.



(i) (a) By considering the addition of two travelling transverse plain waves going in opposite directions along the string:

$$y_+(x, t) = A\cos(\omega t - kx + \phi)$$

$$y_-(x, t) = -A\cos(\omega t + kx + \phi)$$

show that the guitar string will support standing waves of wavelength:

$$\lambda_n = 2\,(L - md)\big/n$$

where n is a positive integer and m is the number of the fret where the musicians finger is placed ($m = 0$ if the string is not held down on any fret). [5 marks]

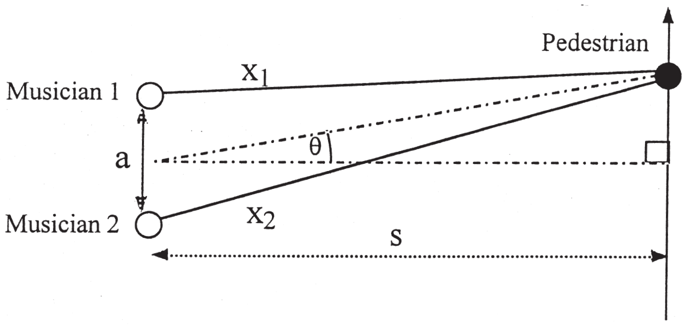
(b) The phase speed of a transverse wave on a string is given by:

$$v = \sqrt{T * /\sigma}$$

where σ is the mass per unit length of the string. If the musician wishes the fundamental frequency ($n = 1$) to be equal to 1 kHz when no frets are used ($m = 0$), what tension T^* needs to be applied to the string? Take $\sigma = 10^{-3}\,\mathrm{kgm}^{-1}$ and $L = 1$ m. [2 marks]

(c) A second identical string on the same guitar is adjusted to the same T^*. The musician then plays the first and second string simultaneously with his finger on no fret ($m = 0$) on the first string and fret $m = 2$ on the second string. This produces beats. What is their frequency ν_{beat} if $d = 0.05$ m. [2 marks]

(ii) (a) The musician goes to the local town square and tries to make some money by playing his guitar. A second musician turns up and stands a distance a away from the first. They both tune their guitars by playing the same note at 1 kHz. A pedestrian walks parallel to them along the other side of the square a distance s away. As he walks the sound intensity from the guitars varies with his position.



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Show that the sound intensity will reach a maximum when:

$$\sin \theta_{\max} = \frac{pv}{a\nu}$$

where p is an integer, v is the speed of sound in air and ν is the wave frequency.
[3 marks]

(b) Using this, calculate the distance Δy the pedestrian walks between maximum points of sound intensity. Let $s = 50\,\mathrm{m}$, $a = 5\,\mathrm{m}$ and $v = 344\,\mathrm{m\,s^{-1}}$.
[3 marks]

(c) Each guitar emits an average power of 1 W. Assume that the sound wave moves out in a hemisphere (does not penetrate the ground). What is the maximum intensity the pedestrian will hear as he walks down the road?
[3 marks]

[TOTAL 18 marks]

SECTION D

8. An atom of mass m is joined to a large molecule by a chemical bond that acts like a spring with spring constant s. The ground-state wave function $\phi_0(x)$ of the centre of mass of the atom satisfies the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\phi_0(x)}{dx^2} + \frac{1}{2}sx^2\phi_0(x) \;=\; E_0\phi_0(x)\ ,$$

where x is the displacement of the atom from its equilibrium position.

(i) Verify that the trial solution

$$\phi_0(x) \;=\; \frac{(ms)^{1/8}}{(\pi\hbar)^{1/4}}\exp\left(-\frac{\sqrt{ms}}{2\hbar}x^2\right)$$

satisfies the Schrödinger equation and determine the corresponding eigenvalue E_0.

[5 marks]

(ii) Show that $\phi_0(x)$ is normalised.

[Throughout this question, the integral

$$\int_{-\infty}^{\infty} x^{2n}e^{-\beta x^2}\,dx \;=\; \frac{(2n)!}{(4\beta)^n n!}\sqrt{\frac{\pi}{\beta}}\ , \qquad n = 0,\ 1,\ 2,\ \ldots$$

may be used without proof.]

[4 marks]

(iii) A measurement is made of the position of the particle. Assuming that the wave function before the measurement was $\phi_0(x)$, where is the particle most likely to be found?

[3 marks]

(iv) Find $\langle x\rangle$ and $\langle x^2\rangle$ when the wave function is $\phi_0(x)$. Show that the root-mean-square width of the corresponding probability density function is

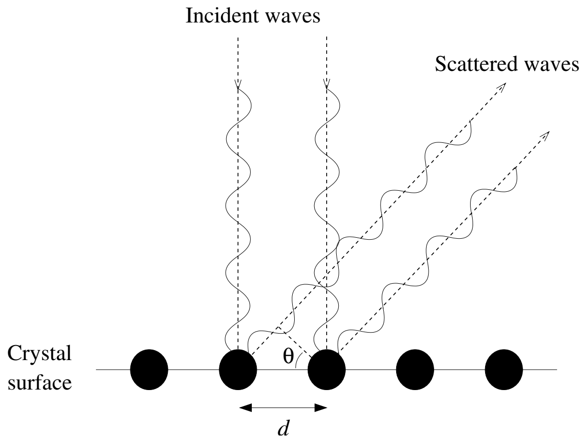
$$\Delta x \;=\; \left(\frac{\hbar^2}{4ms}\right)^{1/4}\ ,$$

and hence obtain a lower bound for the root-mean-square uncertainty in the momentum of the atom.

[6 marks]

[TOTAL 18 marks]

9. A beam of helium atoms (atomic weight 4) of speed v is directed at a crystal of nickel. The incident helium atoms arrive moving at right angles to the surface.



(i) Find the speed v (in $\mathrm{m\,s^{-1}}$) of a helium atom with De Broglie wavelength λ equal to $0.3d$, where d=0.215 nm is the spacing between the nickel atoms on the surface. Show that the kinetic energy of this atom is approximately 0.05 eV. [4 marks]

(ii) Calculate the temperature T at which the root mean square speed of the atoms in a container of helium gas at thermal equilibrium is equal to v. [4 marks]

(iii) Write down an expression for the path difference between waves scattered from adjacent nickel atoms and hence derive the condition for constructive interference. Assuming that $\lambda = 0.3d$, show that the two smallest angles θ (excluding θ=0) at which diffracted beams of helium atoms are observed are 17.5° and 36.9°. [5 marks]

(iv) Show that the angular width $\Delta\theta_n$ of the n^{th} diffracted beam is given approximately by

$$\Delta\theta_n \approx \frac{n\lambda\Delta v}{vd\cos\theta_n},$$

where Δv is the range of atomic speeds in the incident beam. Calculate $\Delta\theta_1$ (in degrees) for an experiment in which $\lambda = 0.3d$ and $\Delta v/v = 0.01$. [5 marks]

[TOTAL 18 marks]

End