UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2006

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

STRUCTURE OF MATTER,VIBRATION & WAVES and QUANTUM PHYSICS

For First-Year Physics Students

Wednesday 7th June 2006: 10.00 to 13.00

Answer ALL parts of Section A, ONE question from Section B, ONE question from Section C and ONE question from Section D.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the SIX answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in SIX answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

SECTION A (Compulsory)

1. (i) An ideal gas undergoes an isothermal compression to one half of its original volume. Sketch this process on a PV diagram, and calculate the value of the work done on the gas if the initial pressure and volume are 10^5 Pa and $0.1\,\mathrm{m}^3$ respectively.

[3 marks]

(ii) The interaction between a pair of atoms is often approximated by the Lennard-Jones 6-12 potential energy, which has the following form:

$$U(r) = \epsilon \left[\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^{6} \right]$$

where r is the separation of the atoms (i.e., the distance between their centres). Show that the equilibrium separation is r_0, and sketch a graph of $U(r)$, indicating clearly the position of the equilibrium separation and the value of U at the minimum.

[3 marks]

(iii) The velocity component distribution function has the form

$$f(v_x) = A\mathrm{e}^{-mv_x^2/2k_BT} \; .$$

Sketch the form of $f(v_x)$ for two different temperatures, clearly indicating which is at the higher temperature.

[3 marks]

[TOTAL 9 marks]

2. (i) Sketch the variation of the displacement with time of a damped harmonic oscillator in the heavily damped, critically damped and lightly damped regimes. The angular frequency of a lightly damped mechanical oscillator is given by:

$$\omega' = \sqrt{\left(\frac{s}{m} - \frac{r^2}{4m^2}\right)}$$

If $s = 2000\,\mathrm{N/m}$ and $m = 2\,\mathrm{kg}$, what value of mechanical resistance r is needed to make the oscillator critically damped?

[3 marks]

(ii) Deep-water surface gravity waves have a phase velocity:

$$v = \sqrt{g/k}$$

where $g = 10\,\mathrm{m\,s^{-2}}$. What type of dispersion is occurring? What is the group velocity v_g? A surfer wishes to catch a wave with wavelength 20 m. Would the surfer need to paddle or be towed by a jet-ski to catch the wave?

[3 marks]

(iii) A mechanical oscillator of mass $m = 0.17\,\mathrm{kg}$ obeys the Equation of Motion:

$$m\frac{d^2x}{dt^2} = -45.7x$$

where all quantities are in SI units. Sketch the variation of the potential energy PE, the kinetic energy KE and total energy TE with position x. If the amplitude of oscillation is 0.034 m, what is the numerical value of TE? What is the numerical value of the maximum velocity of the oscillator?

[3 marks]

[TOTAL 9 marks]

3. (i) In a photoelectric effect experiment using a copper cathode, the stopping potential is 5V when the photon wavelength is 1.28×10^{-7} m. Find the work function of copper in eV.

[3 marks]

(ii) A non-relativistic particle of mass m has kinetic energy $E = \frac{1}{2}mv^2$. Starting from this equation, and using the Planck and De Broglie equations, derive the dispersion relation for non-relativistic particle-waves.

[3 marks]

(iii) The wavelengths of the spectral lines of atomic Hydrogen are given by the Rydberg formula

$$\frac{1}{\lambda_{mn}} = R_H \left(\frac{1}{n^2} - \frac{1}{m^2} \right) ,$$

where m and n $(< m)$ are positive integers and $R_H = 1.097 \times 10^7\,\mathrm{m}^{-1}$. Explain how this result follows from the assumption that the energy levels of a Hydrogen atom are

$$E_n = -\frac{13.6\,\mathrm{eV}}{n^2} , \qquad n = 1, 2, 3, \ldots .$$

[3 marks]

[TOTAL 9 marks]

SECTION B

4. (i) Use the theorem of equipartition of energy to write down an expression for U, the internal energy, of a monatomic ideal gas of N molecules at temperature T.

[2 marks]

(ii) Use the first law of thermodynamics and the expression for U from part (i) to show that the infinitesimal heat flow into a monatomic ideal gas is given by

$$dQ = \frac{3}{2} N k_B dT + P dV \ .$$

Hence, show that C_V, the constant volume heat capacity, of such a gas is given by

$$C_V = \frac{3}{2} N k_B \ .$$

[5 marks]

(iii) A van der Waals gas satisfies the following equations [DO NOT PROVE]:

$$\left(P + a\frac{N^2}{V^2}\right)(V - bN) = N k_B T \quad , \qquad U = \frac{3}{2} N k_B T - a\frac{N^2}{V} \ .$$

Show that in a van der Waals gas

$$dQ = \frac{3}{2} N k_B dT + \frac{N k_B T}{(V - bN)} dV$$

and, hence, that the equation for C_V in an ideal gas, found in part (ii), also applies in a van der Waals gas.

[5 marks]

(iv) State the answers to the following questions:

(a) Is the constant volume heat capacity, C_V, of a diatomic gas: greater than, equal to, or, less than, $\frac{3}{2} N k_B T$?

(b) Is the constant pressure heat capacity, C_P, of a van der Waals gas: greater than, equal to, or, less than, $\frac{3}{2} N k_B T$?

Briefly justify your answers, but DO NOT ATTEMPT DETAILED CALCULATIONS.

[6 marks]

[TOTAL 18 marks]

5. (i) By considering the forces acting on a thin slab of water of thickness dz show that $P(z)$, the pressure of the sea at depth z below the surface, satisfies the following differential equation:

$$\frac{dP}{dz} = \rho_w g$$

where ρ_w is the density (in $\mathrm{kg\,m^{-3}}$) of sea water, and g is the acceleration due to gravity. Hence, assuming that ρ_w is independent of depth, show that $P(z)$ has the form $P(z) = P_0 + \rho_w g z$. What is the constant P_0? [5 marks]

(ii) Use Archimedes' Principle to show that the upward force on a spherical bubble (radius r) of an ideal gas in the sea is

$$F_{up} = \frac{4}{3}\pi r^3 g \left(\rho_w - \frac{m P_{in}}{k_B T}\right)$$

where m is the mass of a single gas molecule and P_{in} is the pressure of the gas in the bubble, and T is its temperature. [3 marks]

(iii) The energy required to increase the area of a liquid surface by dA is γdA, where γ is the surface tension [DO NOT PROVE]. Show that the energy needed to increase the radius of a bubble from r to $r + dr$, in a liquid of surface tension γ, is $8\pi\gamma r dr$. [3 marks]

(iv) Surface tension produces a force F_c on the surface of a bubble directed radially inwards towards the centre. The work required to overcome this force and increase the radius of the bubble by dr is $F_c dr$. Use the result from part (iii) to show that the pressure inside a bubble of radius r is:

$$P_{in} = P_{out} + \frac{2\gamma}{r}$$

where P_{out} is the pressure of the liquid outside the bubble. [4 marks]

(v) A vent on the sea bed at a depth of $0.1\,\mathrm{km}$ discharges bubbles of CO_2 of radius $10\ \mu\mathrm{m}$. Use the data below to calculate:

(a) the pressure of the sea water at this depth,

(b) the pressure of the gas inside the bubble, and,

(c) the net upward force on the bubble.

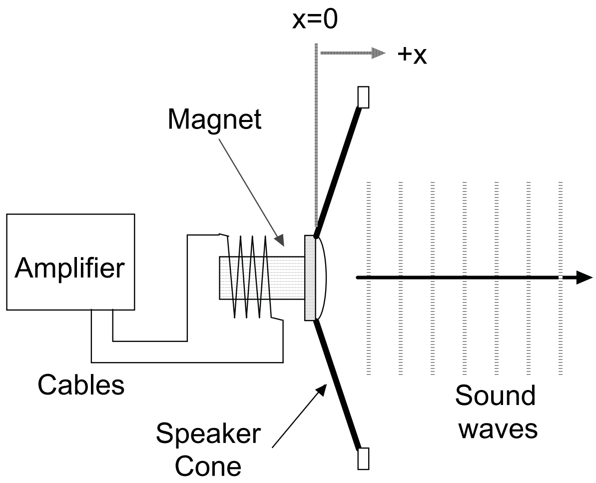
[Pressure at the surface: $P_0 = 1$ atmosphere $= 1.01 \times 10^5\,\mathrm{Pa}$.
Acceleration due to gravity: $g = 9.81\,\mathrm{m\,s^{-2}}$.
Density of sea water: $\rho_w = 1.02 \times 10^3\,\mathrm{kg\,m^3}$.
Mass of CO_2 molecule: $m = 7.31 \times 10^{-26}\,\mathrm{kg}$.
Surface tension of sea water: $\gamma = 0.072\,\mathrm{N\,m^{-1}}$.
Temperature of sea water at a depth of $1\,\mathrm{km}$: $T = 10^\circ\,\mathrm{C}$.]

[3 marks]

[TOTAL 18 marks]

SECTION C

6. A loudspeaker consists of a magnet of mass m_m attached to a cone of mass m_c. The speaker cone offers a restoring force to any displacement away from equilibrium in the x-direction equal to $-Sx$. The speaker cone also offers a mechanical resistive force for motion in the x-direction equal to $-r_c$ multiplied by the velocity. In addition, the volume of air displaced also offers a mechanical resistive force for motion in the x-direction equal to $-r_a A$ multiplied by the velocity, where A is the area of the speaker and r_a is a constant. A HiFi amplifier connected by a cable to a coil of wire surrounding the magnet drives the speaker.



By supplying an AC current of magnitude I_0 at angular frequency ω, the speaker cone and magnet have a force exerted on them in the x-direction equal to $KI_0\cos(\omega t)$, where K is a constant. The Equation of Motion for the displacement of the speaker in the x-direction is therefore in the complex notation:

$$(m_m + m_c)\frac{d^2\underline{x}}{dt^2} = -(r_c + r_a A)\frac{d\underline{x}}{dt} - S\underline{x} + KI_0\exp(j\omega t)$$

Let $m_m = 0.03\,\text{kg}$, $m_c = 0.02\,\text{kg}$, $S = 79,000\,\text{N/m}$, $r_c = 56.82\,\text{Ns/m}$, $A = 0.15\,\text{m}^2$ and $r_a = 40\,\text{Ns/m}^3$.

(i) (a) What is the natural angular frequency ω_0 of oscillation of the speaker? [1 mark]

(b) Using the substitutions $m = (m_m + m_c)$ and $r = (r_c + r_a A)$ or otherwise, show that the Equation of Motion has a steady-state solution of the form:

$$\underline{x} = \underline{B}\exp(j\omega t)$$

where:

$$\underline{B} = -jKI_0\exp(-j\phi)\big/\omega Z_m$$

and Z_m is the magnitude of the mechanical impedance given by:

$$Z_m = \sqrt{\left[(r_c + r_a A)^2 + \left((m_m + m_c)\omega - S/\omega\right)^2\right]}.$$

[4 marks]

(c) Find the real part of the solution. [2 marks]

(ii) (a) Consider the amplitude of oscillation in part (i)(b). By rearranging and differentiating part of the denominator, show that the angular frequency of maximum amplitude, the resonant angular frequency, is given by:

$$\omega_r = \sqrt{\frac{S}{(m_m + m_c)} - \frac{(r_c + r_a A)^2}{2(m_m + m_c)^2}}.$$

[4 marks]

(b) Sketch the variation of the amplitude of oscillation with ω. Show how this varies with $(r_c + r_a A)$. Indicate the position of both the resonant angular frequency ω_r and the natural frequency of oscillation ω_0. [2 marks]

(iii) (a) Sketch the variation of the power absorbed by the speaker from the amplifier with ω. Show how this varies with $(r_c + r_a A)$. [2 marks]

(b) Calculate the spread of frequencies (in Hz) which can usefully be produced by the speaker. What is the difference between the frequency of maximum power transfer and the frequency of maximum amplitude?

[3 marks]

[TOTAL 18 marks]

7. (i) Some marine biologists in a boat are using sonar to study the behaviour of whales under the water. Longitudinal sound waves of a single frequency are emitted from the sonar apparatus under the boat. The sound waves are emitted in bursts of 10^3–10^4 cycles. The sound waves bounce off any whales and return to the sonar apparatus. The relative location and distance of any whales can then be measured by the position of the echo and the delay between the emission and return of each sound wave burst.

(a) Longitudinal sound waves in water have the form:

$$\psi(x,t) = A\cos(\omega t - kx + \phi).$$

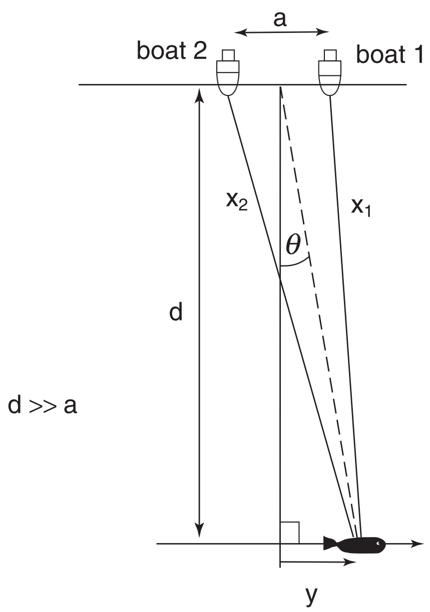
Show that they obey the wave equation:

$$\frac{d^2\psi}{dx^2} = \frac{\rho}{B}\frac{d^2\psi}{dt^2}$$

where B is the bulk modulus and ρ the density of water. What is the phase velocity v of the waves? [4 marks]

(b) A whale swims horizontally at a depth of 600 m below the surface of the sea. However, at a depth of 400 m the water suddenly becomes more salty and changes density. Above 400 m, $B = 2.18 \times 10^9$ Pa and $\rho = 1.05 \times 10^3$ kg m^{-3}. Below 400 m, $B = 2.18 \times 10^9$ Pa and $\rho = 1.2 \times 10^3$ kg m^{-3}. The sonar detects the whale to be at an angle 30° to the vertical below the boat. At what horizontal distance is the whale from the boat? [4 marks]

(ii) The whale swims upwards above the dense, salty water. It then swims horizontally at a constant depth d of 350 m. The marine biologists decide to get a more accurate position of the whale by using sonar from a second boat (boat 2). This boat is at a distance a of 20 m from the first boat (boat 1). The depth of the whale $d \gg a$. The sonar on boat 1 and boat 2 both emit bursts of sound waves of the same frequency f. Assume that these reach the whale simultaneously.



(a) Let the sound wave from boat 1 have the form:

$$\psi_1(x_1, t) = A\cos(\omega t - kx_1)$$

and the sound wave from boat 2 have the form:

$$\psi_2(x_2, t) = A\cos(\omega t - kx_2 + \phi)$$

where ϕ is the phase difference between them. By considering the path lengths x_1 and x_2, show that the two bursts of sound waves undergo constructive interference at depth d according to:

$$y_{\max} = \frac{d\lambda(n + \phi/2\pi)}{a}$$

where n is an integer and y_{max} is a position of maximum constructive interference.

[4 marks]

(b) Suppose the whale is 20 m long. How large should the sonar frequency f be so that the whale can always be detected? Assume $B = 2.18 \times 10^9$ Pa and $\rho = 1.05 \times 10^3\,\mathrm{kg\,m^{-3}}$.

[2 marks]

(c) Consider the addition of $\psi_1(x_1, t)$ and $\psi_2(x_2, t)$ where $\phi = 0$. Using the substitution:

$$X \equiv (x_2 + x_1)/2$$

show that the intensity of the interference pattern at the depth of the whale varies according to:

$$I = 4I_0\cos^2\left(\frac{\pi a y}{d\lambda}\right).$$

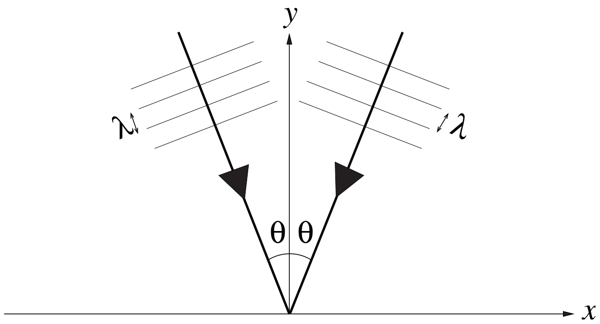
If the sound waves from the sonar apparatus have an amplitude of 0.01 m at a distance of 1 m, what is the amplitude of the interference pattern maxima immediately below the two boats?

[4 marks]

[TOTAL 18 marks]

SECTION D

8. Two laser beams of wavelength $\lambda = 780\,\text{nm}$, each moving at angle $\theta = 15.1^{\circ}$ to the y axis, overlap to create an interference pattern.



(i) The complex representations of the two laser beams are

$$\begin{aligned}\psi_1(x, y) &= A \exp\left(-iky\cos\theta + ikx\sin\theta\right) \;, \\ \psi_2(x, y) &= A \exp\left(-iky\cos\theta - ikx\sin\theta\right) \;,\end{aligned}$$

where $k = 2\pi/\lambda$. Show that the intensity $I(x, y)$ of the interference pattern is $4|A|^2\cos^2(kx\sin\theta)$.

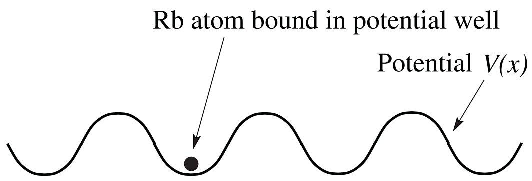
[6 marks]

(ii) Using a multi-angle trigonometic identity, or otherwise, show that the wavelength λ' of the *intensity* variation along the x axis is

$$\lambda' = \frac{\lambda}{2\sin\theta} \;.$$

[4 marks]

A Rb atom in the region of the interference pattern feels the intensity variation as an effective potential, $V(x) = U\cos(k'x)$, where $k' = 2\pi/\lambda'$ and $U = 10^{-10}$ eV. If this potential is strong enough, the Rb atom can become bound (stuck) in one of the potential wells.



(iii) Assuming that a Rb atom of atomic mass 87 is just bound, so that $\Delta x \approx \lambda'/2$, use the Heisenberg uncertainty principle to estimate the momentum uncertainty Δp_x. Hence obtain an estimate of the zero-point kinetic energy of the bound state.

[6 marks]

(iv) How large is the zero-point kinetic energy compared to the depth of the potential well? Would you expect the atom to be bound at zero temperature or not?

[2 marks]

[TOTAL 18 marks]

9. (i) A particle of mass m moves in a simple harmonic potential $V(x) = \frac{1}{2}sx^2$, where s is the effective spring constant. Write down the time-independent Schrödinger equation satisfied by the particle's energy eigenfunctions. [3 marks]

(ii) The ground-state wave function of a simple harmonic oscillator takes the form $\psi_0(x) = \exp(-\alpha x^2)$, where α is a constant. Show that

$$\frac{d^2\psi_0}{dx^2} = (4\alpha^2 x^2 - 2\alpha)\psi_0(x) \ .$$

Find the positive value of α for which $\psi_0(x)$ satisfies the Schrödinger equation from part (i). Hence show that the ground-state energy of the oscillator is $E_0 = \frac{1}{2}\hbar\omega$, where $\omega = \sqrt{s/m}$. [6 marks]

(iii) A carbon atom (atomic mass 12) in a diamond sits in a potential well of effective spring constant $s = 74\,\text{eV}\,\text{Å}^{-2}$. Evaluate the angular frequency of vibration ω. [3 marks]

(iv) Each C atom vibrates about its equilibrium position in three different directions and so acts like three independent oscillators. If these oscillators are treated using classical physics, the principle of equipartition tells us that the heat capacity per atom is $3k_B$. Estimate the heat capacity of 1 kg of diamonds. [3 marks]

(v) The measured heat capacity of diamond at room temperature is about $500\,\text{J kg}^{-1}\,\text{K}^{-1}$, which is a good deal smaller than the result you should have obtained in part (iv). Why?

[Hint: compare the values of $\hbar\omega$ and k_BT.] [3 marks]

[TOTAL 18 marks]

End