UNIVERSITY OF LONDON

BSc/MSci EXAMINATION June 2005

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

STRUCTURE OF MATTER, VIBRATION & WAVES and QUANTUM PHYSICS

For First-Year Physics Students

Wednesday 8th June 2005: 10.00 to 13.00

Answer ALL parts of Section A, ONE question from Section B, ONE question from Section C and ONE question from Section D.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the SIX answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in SIX answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

Values of Constants

Boltzmann's constant	k_B	$=$	$1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Acceleration due to gravity	g	$=$	$9.81\,\mathrm{m\,s^{-2}}$
Absolute zero of temperature	$0\,\mathrm{K}$	$=$	$-273^\circ\,\mathrm{C}$
Planck's constant	h	$=$	$6.63 \times 10^{-34}\,\mathrm{J\,s}$
Electron mass	m_e	$=$	$9.11 \times 10^{-31}\,\mathrm{kg}$
Speed of light	c	$=$	$3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$

SECTION A (Compulsory)

1. (i) In a quasistaic adiabatic process in a monatomic ideal gas $PV^{5/3}$ = constant [DO NOT PROVE]. A monatomic ideal gas initially has a pressure of P_0 and a volume of V_0. It undergoes a quasistatic adiabatic compression to half its initial volume. Show that the work done on the gas is

$$W = \frac{3}{2} P_0 V_0 \left(2^{2/3} - 1\right) \ .$$

[3 marks]

(ii) Ice of density $920\,\mathrm{kg\ m^{-3}}$ is 90% submerged when floating in seawater. Calculate the minimum volume of ice which would just support a 30 kg penguin (i.e., keep the penguin's feet out of the water).

[3 marks]

(iii) A steel cube has sides of length 0.1 m. Its temperature is raised by 20 K. Calculate:

(a) the increase in the length of each side, and

(b) the increase in the volume of the cube.

The linear expansion coefficient of steel is $1.2 \times 10^{-5}\,\mathrm{K^{-1}}$.

[3 marks]

[TOTAL 9 marks]

2. (i) Show that $\psi(x, t) = 0.3 \sin(532t - 6.98x + 0.713)$ is a solution of the wave equation:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

All quantities are in SI units. What is the phase velocity v of the wave? [3 marks]

(ii) The variation of the displacement x with time t of a mechanical oscillator in the lightly damped regime is given by:

$$x(t) = 7.3 \exp(-0.84t) \cos(\omega' t)$$

where:

$$\omega' = \sqrt{\left(76.3 - \frac{r^2}{4m^2}\right)}$$

r is the mechanical resistance and m the mass. All quantities are in SI units. Sketch the variation of the displacement with time. Let $m = 5\,\text{kg}$. What is the restoring force spring constant s? How does the total energy $TE(t)$ of the oscillator vary with time? What value of r is required to make the oscillator critically damped? [3 marks]

(iii) Ripples on the surface of a bath filled with water have a phase velocity given by:

$$v = \sqrt{\frac{\sigma k}{\rho}}$$

where σ is the surface tension and ρ is the density of water. Is the dispersion normal or anomalous? Calculate the ratio between the phase velocity v and the group velocity v_g of the ripples. If $\sigma = 0.073\,\text{Nm}^{-1}$ and $\rho = 10^3\,\text{kgm}^{-3}$, how long would it take ripples of $\lambda = 0.01\,\text{m}$ to travel the length of a 1.5 m long bath? [3 marks]

[TOTAL 9 marks]

3. (i) A coherent beam of light of wavevector k passes through two narrow parallel slits separated by a distance d. The *amplitude* of the light emerging at angle θ is proportional to

$$1 + e^{ikd\sin\theta} \; .$$

Show that the *intensity* of the light emerging at angle θ is proportional to $1 + \cos(kd\sin\theta)$.

[3 marks]

(ii) A photon of wavelenth 500 nm Compton scatters from a stationary electron. Given that the scattering angle is 50° and that the Compton scattering formula is

$$\lambda_f - \lambda_i = \frac{h}{mc}(1 - \cos\theta) \; ,$$

evaluate the energy (in eV) transferred to the electron.

[3 marks]

(iii) An electron microscope cannot resolve objects smaller than the de Broglie wavelength of the electrons (in practice the resolution is much worse than this). Using the relativistic relationship,

$$(K + mc^2)^2 = m^2c^4 + p^2c^2 \; ,$$

between the kinetic energy K, rest mass m, and momentum p of the electrons, estimate the best possible resolution of an electron microscope operating with an electron kinetic energy of 300 keV.

[3 marks]

[TOTAL 9 marks]

SECTION B

4. (i) The probability that a molecule in a gas has a speed between v and $v + \mathrm{d}v$ is given by the Maxwell-Boltzmann speed distribution:

$$f(v)\mathrm{d}v = Av^2 \mathrm{e}^{-\alpha v^2} \mathrm{d}v$$

where $\alpha = m/2k_B T$, m is the mass of one molecule and A is a constant. Sketch a graph of $f(v)$, indicating v_{mp}, the most probable speed. Show that

$$v_{mp} = \left(\frac{2k_B T}{m}\right)^{1/2} .$$

[5 marks]

(ii) Using the following standard integral (which you may assume without proof)

$$\int_0^{+\infty} x^2 \mathrm{e}^{-\alpha x^2} \, \mathrm{d}x = \frac{1}{4}\left(\frac{\pi}{\alpha^3}\right)^{1/2}$$

show that

$$A = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} .$$

[5 marks]

(iii) Using the following standard integral (which you may assume without proof)

$$\int_0^{\infty} x^4 e^{-\alpha x^2} \, dx = \frac{3}{8}\left(\frac{\pi}{\alpha^5}\right)^{1/2}$$

show that the average kinetic energy of translational motion, per molecule is $\frac{3}{2} k_B T$.

[5 marks]

(iv) Explain, briefly and qualitatively, why the expression derived in the previous part cannot be used to calculate the average energy of a diatomic molecule.

[3 marks]

[TOTAL 18 marks]

5. (i) The various phases of a substance can be shown on a PV diagram. For the restricted range of pressures and volumes over which only the liquid and gas phases occur, sketch a PV diagram for a typical substance, showing clearly:

(a) the liquid and gas regions,

(b) the region in which the two phases can coexist,

(c) the critical point,

(d) an isotherm corresponding to $T < T_c$, where T_c is the critical temperature, and

(e) an isotherm for $T > T_c$.

[8 marks]

(ii) The gas and liquid phases are often modelled using the van der Waals equation of state:

$$\left(P + \frac{aN^2}{V^2}\right)(V - bN) = Nk_BT \quad [\text{DO NOT PROVE}]$$

where N is the number of molecules in volume V, and a and b are constants. The conditions at the critical point can be found in terms of a and b by assuming that if T is held fixed at T_c (i.e., along the $T = T_c$ isotherm) then at the critical point $\dfrac{\mathrm{d}P}{\mathrm{d}V} = \dfrac{\mathrm{d}^2P}{\mathrm{d}V^2} = 0$. Use this assumption to show that:

$$V_c = 3Nb \qquad \text{and} \qquad T_c = \frac{8a}{27k_Bb}$$

where V_c is the volume at the critical point.

[6 marks]

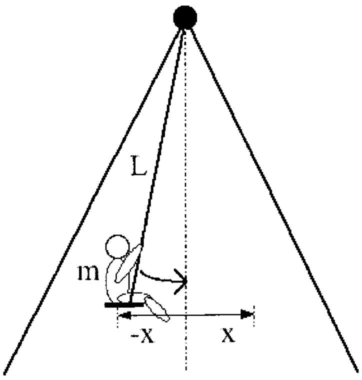
(iii) For Nitrogen $T_c = 126\,\mathrm{K}$ and $a = 3.86 \times 10^{-49}\,\mathrm{J\,m^3}$. Given that b is approximately the volume of one molecule, and assuming that the molecule is spherical, estimate the radius of a Nitrogen molecule.

[4 marks]

[TOTAL 18 marks]

SECTION C

6.



A child of mass m plays on a swing of cable length L. This can be treated as a simple pendulum moving in the x direction. Air resistance and mechanical friction produce a mechanical resistance r. By moving the upper part of her body and her lower legs backwards and forwards she can change her centre of mass in the x direction by a small distance d. By doing this at angular frequency ω she produces a periodic driving force $F_0 \cos(\omega t)$ where:

$$F_0 = \frac{mg}{L} d\,.$$

In the complex notation this results in an Equation of Motion:

$$m\frac{d^2\underline{x}}{dt^2} = -r\frac{d\underline{x}}{dt} - \frac{mg}{L}\underline{x} + \frac{mgd}{L}\exp(j\omega t)$$

where g is the acceleration due to gravity. Let $g = 10\,\mathrm{ms}^{-2}$, $m = 20\,\mathrm{kg}$, $L = 2.5\,\mathrm{m}$, $r = 10\,\mathrm{Nsm}^{-1}$, $d = 0.2\,\mathrm{m}$.

(i) What is the numerical value of the natural frequency of oscillation ω_0 of the swing? [2 marks]

(ii) Show that the Equation of Motion has a steady-state solution of the form:

$$\underline{x} = \underline{A}\exp(j\omega t)$$

where:

$$\underline{A} = -j\frac{mgd}{L}\exp(-j\phi) \Big/ \omega Z_m$$

where Z_m, the magnitude of the mechanical impedance, is given by:

$$Z_m = \sqrt{\left[r^2 + m^2(\omega - g/L\omega)^2\right]}\,.$$

[5 marks]

(iii) Find the real part of the solution.

[2 marks]

(iv) The amplitude of the swing's oscillation varies with driving force angular frequency ω. The phase of the swing's oscillation relative to the driving force also varies with ω. It can be shown that the $\pi/2$ out-of-phase (relative to the driving force) component of the amplitude is given by:

$$x_{out}(t) = \frac{(mgd/L)}{\omega Z_m} \sin(\omega t) \cos(\phi) \, .$$

Find an expression for the instantaneous power $P_{out}(t)$ supplied by the motion of the child's upper body and legs. Hence, show that the average power is given by:

$$P_{av} = \frac{(mgd/L)^2}{2Z_m^2} r \, .$$

You may find the following integral useful:

$$\int_0^T \cos^2(\omega t) dt = \frac{T}{2} \, .$$

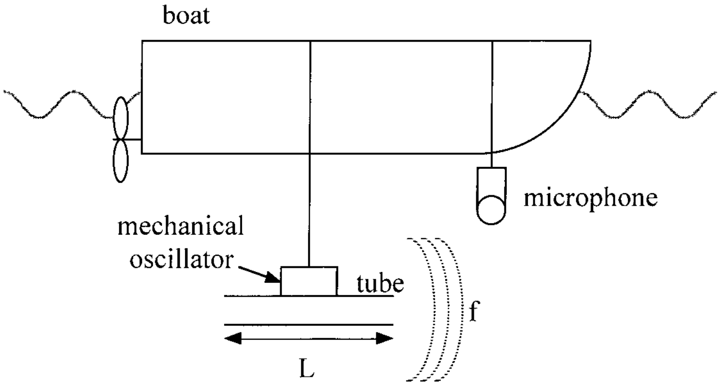
[4 marks]

(v) What is the maximum power at ω_0? What is the amplitude at ω_0? Is this the largest amplitude value possible? Between what range of frequencies f_1 and f_2 (in Hz) can the child move her body to make the swing oscillate?

[5 marks]

[TOTAL 18 marks]

7. (i) A scientist designs a simple underwater sonar system. A metal tube of length L, with both ends open, is lowered into the sea from a boat. A mechanical oscillator is used to create sound waves in the tube. By pulsing the oscillator, the tube can be made to emit sound wave pulses of frequency $f = 2\,\text{kHz}$. A microphone picks up any reflected echoes from objects in the sea.



(a) Sketch the 1st, 2nd, 3rd and 4th modes (ie. harmonics) of a one-dimensional standing wave between $x = 0$ and $x = L$ where the boundary conditions are such that there is an *antinode* at $x = 0$ and an *antinode* at $x = L$. [2 marks]

(b) In the tube, longitudinal sound waves travelling in the positive x-direction $\psi_+(x, t)$ and in the negative x-direction $\psi_-(x, t)$ have the general form:

$$\psi_+(x, t) = A\cos(\omega t - kx + \phi)$$
$$\psi_-(x, t) = A\cos(\omega t + kx + \phi)$$

Let the waves in the tube be reflected at $x = 0$ and $x = L$ according to:

$$\psi_+(x = 0, t) = \psi_-(x = 0, t)$$
$$\psi_+(x = L, t) = \psi_-(x = L, t)$$

By combining $\psi_+(x, t)$ and $\psi_-(x, t)$, show that standing sound waves in the tube have allowed wavelengths given by:

$$\lambda_n = 2L/n$$

where $n = 1, 2, 3, 4, \ldots$. [4 marks]

(c) The phase velocity of sound waves in water is given by:

$$v = \sqrt{B/\rho}$$

where B is the bulk modulus and ρ is the density. If $B = 2.18 \times 10^9$ Pa and $\rho = 10^3\,\text{kgm}^{-3}$, what length of tube L would be required to give a 1st harmonic equal to f? [2 marks]

(ii) (a) A shark swims towards the boat at a velocity of $u_{Shark} = 1\,\mathrm{ms}^{-1}$. The boat is stationary in the water. A pulse from the sonar of wavelength λ and frequency f reaches the shark. By considering the compression of the wavelength as the shark swims into it, show that the frequency f' received by the shark is given by:

$$f' = \frac{(v + u_{Shark})f}{v}$$

where v is the phase velocity of the sound waves given above by the equation in part (i)(c). [4 marks]

(b) The pulse bounces back from the shark to the boat. What frequency f'' is detected by the scientist due to the motion of the shark? [2 marks]

(c) At a depth of 10 m below surface the water suddenly becomes less salty so that its density changes to $0.9 \times 10^3\,\mathrm{kgm}^{-3}$. Beyond what angle to the vertical would the sonar no longer detect anything in this less salty water? The shark is at a depth $< 10\,\mathrm{m}$. The scientist receive two echoes from the shark. What is the minimum horizontal distance the shark has to be from the boat? [2 marks]

(d) The sound wave pulse from the sonar has a total average power of 1 W. Assume that this is emitted in all directions. Let the shark be at a horizontal distance of 20 m and a depth of 5m. Assume that it has an effective surface area of $1\,\mathrm{m}^2$ perpendicular to the direction of travel of the sound wave. What intensity will the microphone detect in the returning echo from the shark? [2 marks]

[TOTAL 18 marks]

SECTION D

8. (i) Sketch the apparatus used by Lenard to observe the photoelectric effect. [3 marks]

(ii) Sketch the measured current as a function of the applied voltage, indicating the saturation current $J_{\max}$ and the threshold voltage $-V_0$ below which the current is zero.

[3 marks]

(iii) Sketch eV_0 as a function of the frequency ν of the incident light. Indicate the work function W and the threshold frequency ν_0 below which no current is measured. What is the observed value of the slope of this graph?

[3 marks]

(iv) Discuss three aspects of the results of the experiment that cannot be understood using classical physics.

[3 marks]

(v) A beam of ultraviolet light of wavelength $\lambda = 124 \times 10^{-9}$ m, intensity $I = 1.6 \times 10^{-12}$ Jm^{-2}s^{-1}, and area $A = 10^{-4}$m falls on a metal of work function $W = 5$ eV.

(a) Obtain a classical estimate of the time delay before the first photoelectron appears by working out how long it takes for W Joules of energy to be accumulated over the area of an atom of radius 10^{-10} m. [3 marks]

(b) In quantum physics, emission can occur as soon as the first photon strikes the surface. Estimate the delay by calculating the average time between the arrival of successive photons.

[3 marks]

[TOTAL 18 marks]

9. The wave function of a particle of mass m confined to the region $0 < x < a$ is

$$\psi(x) = \sqrt{\frac{30}{a^5}}\, x(a-x)\ .$$

(i) Confirm that $\psi(x)$ is normalised. [4 marks]

(ii) Evaluate $\langle x \rangle$. [4 marks]

(iii) Write down the definition of the root-mean-square position uncertainty Δx in terms of $\langle x^2 \rangle$ and $\langle x \rangle$.

Given that $\langle x^2 \rangle = \dfrac{2a^2}{7}$, show that $\Delta x = \dfrac{a}{2\sqrt{7}}$. [3 marks]

(iv) State Heisenberg's uncertainty principle and use it to obtain a strict lower bound on the root-mean-square momentum uncertainty Δp. [3 marks]

(v) Write down the time-independent Schrödinger equation satisfied by the energy eigenfunctions $\phi_n(x)$ of a particle of mass m moving in a potential $V(x)$.

If $\psi(x)$ is an eigenfunction with energy eigenvalue E, show that

$$V(x) = E - \frac{\hbar^2}{mx(a-x)}\ .$$

[4 marks]

[TOTAL 18 marks]

End