First-Year Mathematics

Solutions to Problem Set 9

1. The curl of a vector field

$$\boldsymbol{V} = P(x,y,z)\,\boldsymbol{i} + Q(x,y,z)\,\boldsymbol{j} + R(x,y,z)\,\boldsymbol{k} \tag{1}$$

is given by the determinant

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{V} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}\,. \end{aligned}\tag{2}$$

(a) We identify

$$P = 2xyz^3\,, \qquad Q = x^2y^3 + 2y\,, \qquad R = 3x^2yz^2\,, \tag{3}$$

from which we calculate

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{V} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz^3 & x^2y^3 + 2y & 3x^2yz^2 \end{vmatrix} \\ &= 3x^2z^2\,\boldsymbol{i} + (6xyz^2 - 6xyz^2)\,\boldsymbol{j} + (2xy^3 - 2xz^3)\,\boldsymbol{k} \\ &= 3x^2z^2\,\boldsymbol{i} + 2x(y^3 - z^3)\,\boldsymbol{k}\,, \end{aligned}\tag{4}$$

(b) We identify

$$P = 2xy\,, \qquad Q = x^2 + 2yz\,, \qquad R = y^2 + 1\,, \tag{5}$$

from which we calculate

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{V} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy & x^2 + 2yz & y^2 + 1 \end{vmatrix} \\ &= (2y - 2y)\,\boldsymbol{i} + (0 - 0)\,\boldsymbol{j} + (2x - 2x)\,\boldsymbol{k} \\ &= 0\,, \end{aligned}\tag{6}$$

(c) We identify

$$P = -xy\,, \qquad Q = -yz\,, \qquad R = -xz\,, \tag{7}$$

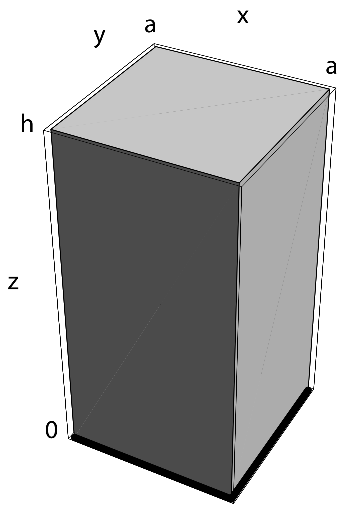
from which we calculate

$$\nabla \times \boldsymbol{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ -xy & -yz & -xz \end{vmatrix}$$

$$= [0 - (-y)]\,\boldsymbol{i} + [0 - (-z)]\,\boldsymbol{j} + [0 - (-x)]\,\boldsymbol{k}$$

$$= y\,\boldsymbol{i} + z\,\boldsymbol{j} + x\,\boldsymbol{k}\,. \qquad (8)$$

2. (a) The square prism and the bounding curve (indicated by the bold line) are shown below:



With $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$, we have that

$$d\boldsymbol{r} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j} + dz\,\boldsymbol{k}\,, \qquad (9)$$

so that

$$\boldsymbol{V} \cdot d\boldsymbol{r} = -xy\,dx - yz\,dy - xz\,dz\,. \qquad (10)$$

The bounding curve lies in x-y plane, so $z = 0$ and $\boldsymbol{V}\cdot d\boldsymbol{r} = -xy\,dx$. The line integral around the bounding curve consists of four straight segments: $(0,0,0) \to (a,0,0)$, $(a,0,0) \to (a,a,0)$, $(a,a,0) \to (0,a,0)$, and $(0,a,0) \to (0,0,0)$. The ranges of x and y along each of these segments are

$$(0,0,0) \to (a,0,0):\ 0 \le x \le a\,, \qquad y = 0\,, \qquad dy = 0; \qquad (11)$$

$$(a,0,0) \to (a,a,0):\ 0 \le y \le a\,, \qquad x = a\,, \qquad dx = 0; \qquad (12)$$

$$(a,a,0) \to (0,a,0):\ a \ge x \ge 0\,, \qquad y = a\,, \qquad dy = 0; \qquad (13)$$

$$(0,a,0) \to (0,0,0):\ a \ge y \ge 0\,, \qquad x = 0\,, \qquad dx = 0. \qquad (14)$$

Thus, the only contribution to the line integral around the bounding curve is along the segment $(a, a, 0) \to (0, a, 0)$, and we obtain

$$\oint_{\partial\sigma} \boldsymbol{V} \cdot d\boldsymbol{r} = -a \int_a^0 x\,dx = \tfrac{1}{2}a^3\,. \tag{15}$$

(b) The curl of $\boldsymbol{V}$ was calculated in Part 1(c):

$$\boldsymbol{\nabla} \times \boldsymbol{V} = y\,\boldsymbol{i} + z\,\boldsymbol{j} + x\,\boldsymbol{k}\,. \tag{16}$$

The sides of the prism are the planes $x = 0$, $x = a$, $y = 0$, and $y = a$, and the top is the plane $z = h$. The corresponding *outward* unit normals (which can be obtained either from the gradient or by inspection) and the "dot" products $(\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n}$ evaluated on the respective planes are

$$x = 0: \quad \boldsymbol{n} = -\boldsymbol{i}; \qquad (\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n} = -y\,, \tag{17}$$

$$x = a: \quad \boldsymbol{n} = \boldsymbol{i}; \qquad (\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n} = y\,, \tag{18}$$

$$y = 0: \quad \boldsymbol{n} = -\boldsymbol{j}; \qquad (\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n} = -z\,, \tag{19}$$

$$y = a: \quad \boldsymbol{n} = \boldsymbol{j}; \qquad (\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n} = z\,, \tag{20}$$

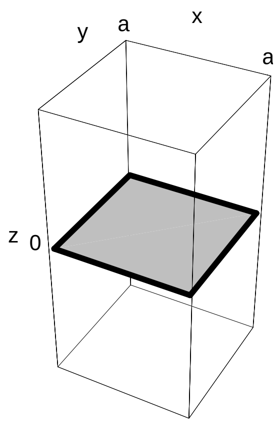
$$z = h: \quad \boldsymbol{n} = \boldsymbol{k}; \qquad (\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n} = x\,. \tag{21}$$

The integrals over $x = 0$ and $x = a$ are seen to cancel, as are those over $y = 0$ and $y = a$, leaving only the integral over $z = h$, which yields

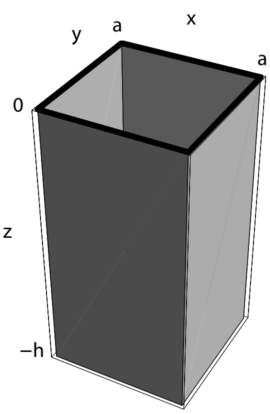
$$\iint_\sigma (\boldsymbol{\nabla} \times \boldsymbol{V}) \cdot \boldsymbol{n}\,d\sigma = \int_0^a x\,dx \int_0^a dy = \tfrac{1}{2}a^2 \times a = \tfrac{1}{2}a^3\,, \tag{22}$$

in agreement with Eq. (15).

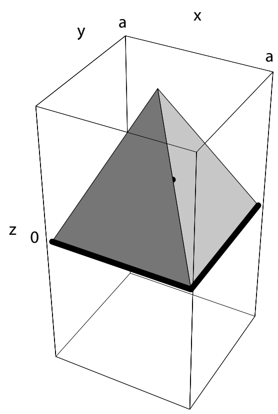
(c) The surfaces in (i), (ii) and (iii) are shown below:



(a)



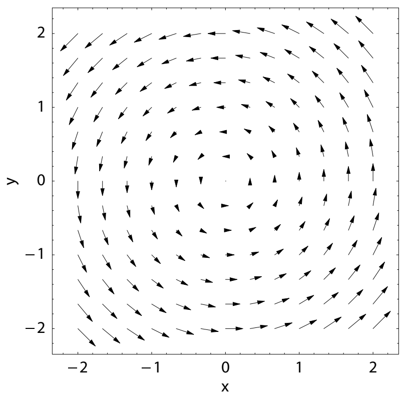
(b)



(c)

The surfaces in (i) and (ii) have the same bounding curve and the same orientation as the square prism above. Thus, the right-hand side of Stokes' theorem is the same as that of the prism. The surface in (ii) has the same bounding curve as the square prism, but the opposite orientation. Since the positive sense of integration is defined according to the right-hand rule, the integral around the bounding curve is taken in the opposite sense to that of the square prism. Hence, the right-hand side of Stokes' theorem has the same magnitude but *opposite* sign as the prism.

3. The vector field $\boldsymbol{V} = -y\,\boldsymbol{i} + x\,\boldsymbol{j}$ is plotted below in the region $-2 \le x \le 2$ and $-2 \le y \le 2$:



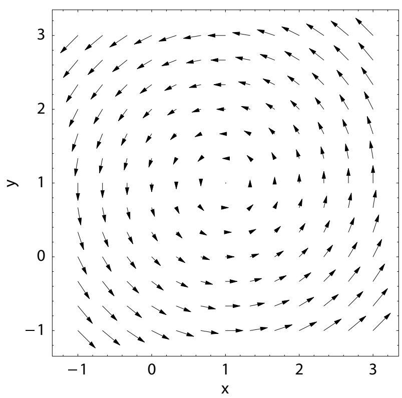
(a) The relative velocity field

$$\boldsymbol{V}(x,y) - \boldsymbol{V}(x_0,y_0) = -(y - y_0)\,\boldsymbol{i} + x - x_0)\,\boldsymbol{j} \tag{23}$$

represents a shift of the origin of $\boldsymbol{V}$. Its curl is

$$\boldsymbol{\nabla} \times [-(y - y_0)\,\boldsymbol{i} + (x - x_0)\,\boldsymbol{j}] = 2\,, \tag{24}$$

as before. The relative velocity field is plotted below for $x_0 = 1$ and $y_0 = 1$ in the region $-1 \le x \le 3$ and $-1 \le y \le 3$:

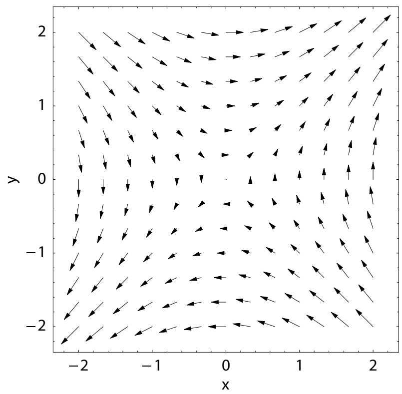


Note that this field is *identical* to the original vector field. The same result would have been obtained had we chosen *any* other point because the curl is independent of position. The fact that the curl is positive is evident from the counter-clockwise "swirl" of the vectors around the origin, just as in the original vector field.

(b) The curl of $\boldsymbol{V}(x, y) = (ax + by)\,\boldsymbol{i} + (cx + dy)\,\boldsymbol{j}$ is

$$\boldsymbol{\nabla} \times \boldsymbol{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ ax + by & cx + dy & 0 \end{vmatrix} = (c - b)\,\boldsymbol{k}\,. \tag{25}$$

Thus, the curl is positive if $c - b > 0$, negative if $c - b < 0$, and vanishes if $c - b = 0$. Notice that these conclusions are valid regardless of the values of a and b because these constants are eliminated upon taking the derivatives to form the curl. The vector field $\boldsymbol{V} = y\,\boldsymbol{i} + x\,\boldsymbol{j}$, which has zero curl everywhere, is shown below in the region $-2 \le x \le 2$ and $-2 \le y \le 2$:

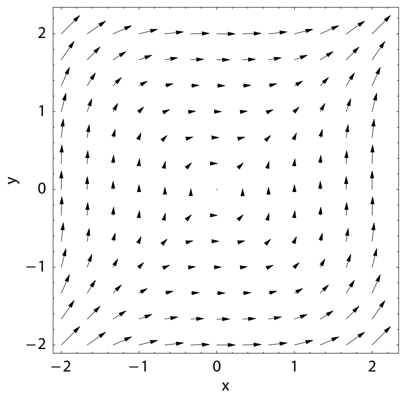


The interpretation of a vector field having zero curl is that there is no net "swirl" about the reference point.

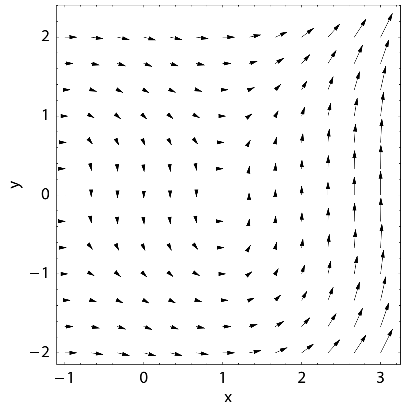
(c) The curl of $\boldsymbol{V} = y^2\,\boldsymbol{i} + x^2\,\boldsymbol{j}$ is

$$\boldsymbol{\nabla} \times \boldsymbol{V} = 2(x - y)\,\boldsymbol{k}\,,$$

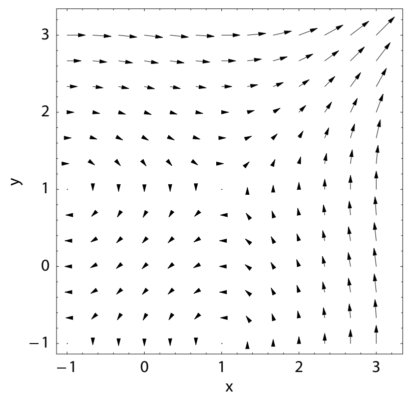
which is now position-dependent and can take positive values, negative values, or vanish. The curl vanishes along the line $y = x$, which includes the origin. The vector field $\boldsymbol{V}(x, y) = y^2\,\boldsymbol{i} + x^2\,\boldsymbol{j}$ is plotted below in the region $-2 \le x \le 2$ and $-2 \le y \le 2$:



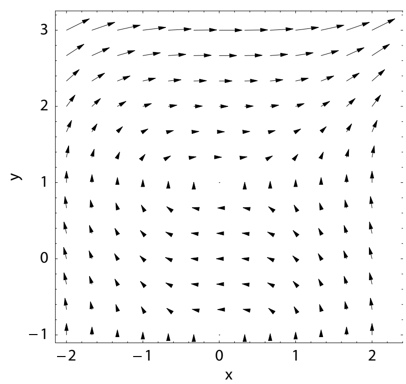
The vanishing curl is seen to result from the counter-clockwise and clockwise “swirls” exactly balancing. The vector field relative to the point $(1, 0)$ is shown below in the region $-1 \leq x \leq 3$ and $-2 \leq y \leq 2$:



The positive curl results from the net counter-clockwise swirl of the vector field about our reference point. The vector field relative to the point $(1, 1)$ is shown below in the region $-1 \leq x \leq 3$ and $-1 \leq y \leq 3$:



The vanishing curl is seen to result from the balancing of the counter-clockwise and clockwise “swirls”. The vector field relative to the point $(0, 1)$ is shown below in the region $-2 \leq x \leq 2$ and $-3 \leq y \leq 1$:



The negative curl results from the net clockwise “swirl” of the vector field about our reference point.