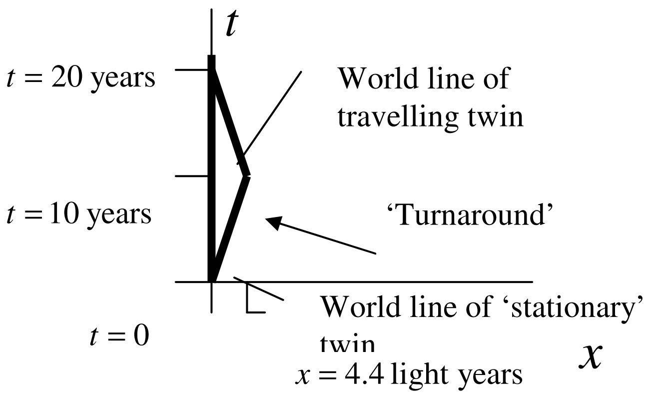
RELATIVITY **Problems 2: Answers** 16Dec04

1. It was shown in problem 2 in the last problem sheet that on the outbound journey, on reaching Alpha Centauri, the spacetraveller records 10 years of his time, whilst his earthbound twin brother experiences 11 years of earth time (i.e. $\gamma = 1.1$). The return journey is exactly similar (except of course that the velocity is reversed – but this doesn't affect γ), so that the travelling twin will return after a total time of 20 years have elapsed according to his clock. The earthbound twin will record his brother's return as being 22 years, and thus the earthbound twin will now be (genuinely!) older than his brother.



The point here is that despite the apparent symmetry between the brothers' observations of each other, the travelling twin experiences an enormous acceleration at the 'turnaround' point in the journey, as is evident from the spacetime diagram drawn here from the earth observer's viewpoint.

2. Use the velocity addition formula $v = \dfrac{u + v'}{1 + uv'/c^2}$ where u is the relative velocity between frames, v' is the known velocity in the 'moving' frame and v is the resultant velocity to be calculated.

(a) An observer on the first particle sees the Lab moving with velocity $u = -0.8c$. The second particle moves with velocity $v' = -0.8c$ in the 'moving' lab frame. So resultant velocity of second particle with respect to first $v = \dfrac{-0.8c - 0.8c}{1 + 0.8^2 c^2/c^2}$ $\rightarrow$ $\underline{v = -0.98c}$.

(b) Now $u = 0.8c$, $v' = 0.8c$, so $\underline{v = +0.98c}$.

(c) The lab observer records the relative velocity of the first particle relative to the second to be $+0.8c - (-0.8c)$ $\underline{= 1.6c}$.

The point is that this is a 'geometrical' velocity recorded by the lab rods and clocks. No material particle, momentum, energy or information is actually being transferred at this rate. The relativistic resultant velocity formula is applied, as in parts (a) and

(b) to situations where the velocity is known in one frame and we want to know the velocity observed *from another frame*. In part (c) only one frame is involved.

3. (a) Lorentz transf. of space-time coordinates in $O \rightarrow$ coordinates in O'

$$x' = \gamma(x - \beta ct)\ ;\ ct' = \gamma(ct - \beta x)$$

where $u = 0.9c \Rightarrow \beta = 0.9$ and $\gamma = (1-\beta^2)^{-1/2}$.

$x'_A = 2.3(0 - 0.9\times 3\times 10^8 \times 100\times 10^{-9})$: $\quad x'_A = -62\,\text{m}$

$t'_A = 2.3(100\times 10^{-9} - 0.9\times 3\times 10^8 \times 0)$: $\quad t'_A = 230\,\text{ns}$

$x'_B = 2.3(-100 - 0.9\times 3\times 10^8 \times 0)$: $\quad x'_B = -230\,\text{m}$

$t'_B = 2.3(0 - 0.9\times(-100))$: $\quad t'_B = 690\,\text{ns}$

(b) From $O' \rightarrow O$ we need the inverse transformations: i.e. $u \rightarrow -u$

$x_C = 2.3(0 + 0.9\times 3\times 10^8 \times 100\times 10^{-9})$: $\quad x_C = +62\,\text{m}$

$t_C = 2.3(100\times 10^{-9} + 0.9\times 0)$: $\quad t_C = 230\,\text{ns}$

$x_D = 2.3(-100 + 0.9\times 3\times 10^8 \times 0)$: $\quad x_D = -230\,\text{m}$

$t_D = 2.3(0 + 0.9\times(-100))$: $\quad t'_B = -690\,\text{ns}$

(c) In O: $p = 0.5$ MeV/c; $m = 0.5$ MeV/c^2

We need $E = \left(p^2c^2 + m^2c^4\right)^{1/2} = \left(0.5^2 + 0.5^2\right)^{1/2} = 0.71\,\text{MeV}$

Lorentz Transf. for $\left(E, p_x\right)$ are identical for those for (t, x), so

$$p'_x = 2.3(0.5 - 0.9\times 0.71) = -0.31\,\text{MeV/c}$$

$$E'/c = 2.3(0.71 - 0.9\times 0.5) = 0.60\,\text{MeV/c}$$

$\rightarrow p' = 0.32\,\text{MeV/c}$ in –ve x-direction; $\quad E' = 0.6\,\text{MeV}$

(d) In O': $K' = 1\text{MeV}$ So $\quad E' = K' + mc^2 = 1 + 0.5 = 1.5\text{MeV}$

and $\quad p'c = (E'^2 - m^2c^4)^{1/2} = [(1.5)^2 - (0.5)^2]^{1/2} = 1.41\text{MeV}$

So $\quad (p'_x, E'/c) = (1.4, 1.5)\,\text{MeV/c}$

In O: We use the inverse transformation: from $O \rightarrow O'$: i.e. $u \rightarrow -u$

$$p_x = 2.3(1.41 + 0.9\times 1.5) = 6.35\,\text{MeV/c}$$

$$E/c = 2.3(1.5 + 0.9\times 1.41) = 6.37\ \text{MeV/c}$$

So $E = 6.37\,\text{MeV}$

4. (a) $m_e = 0.511\times 10^6\, eV/c^2$:

so rest energy $\quad E_{rest} = 0.511\times 10^6\, eV = 0.511\times 10^6 \times 1.6\times 10^{-19}\, J$

and mass $= \dfrac{E_{rest}}{c^2} = \dfrac{0.511\times 1.6\times 10^{-13}}{\left(3\times 10^8\right)^2}$ $\quad \underline{= 0.9\times 10^{-30}\, kg}$.

(b) $E_{rest} = 0.511\times 10^6 \times 1.6\times 10^{-19}\, J$ $\quad \underline{= 0.08\, pJ}$.

(c) Rest energy of electron and positron $= 2m_ec^2$, So energy liberated (see (b) above) $= 2\times 0.08\,pJ$

$= 0.16\,pJ$.

(d) $E = 10^{19}\,eV = 1.6\times 10^{-19}\times 10^{19}$ $= 1.6J$.

5. The flight path λ, is given by *velocity* times *lifetime*, where the lifetime is given in the frame of reference where the flight path is measured (the laboratory frame). Due to time dilation, the lifetime is dilated by the Lorentz factor γ, compared with the *proper* lifetime (as measured in the particle's rest frame). Hence $\lambda = v\times\gamma\times\tau_\pi$

(a) For the pion with velocity $0.1c$, $\gamma = \left(1-(0.1)^2\right)^{-1/2} \approx 1.005$ and $\lambda = 1.005\times 0.1\times 3\times 10^8\times 10^{-16}\,\text{m}$ $= 3nm$.

(b) We need γv for a pion of momentum $1GeV/c$. This is most easily obtained from the expressions in question 8 above

$$v\gamma = \frac{pc}{E}\times\frac{E}{mc^2} = \frac{pc}{mc^2}\ .$$

Hence $\lambda = \left(10^9/135\times 10^6\right)\times 3\times 10^8\times 10^{-16}$ $= 220nm$.

(c) The pion with kinetic energy 1TeV is ultra-relativistic and we can use the photon-like approximation $K = E = pc$.

i.e. $p = 1TeV/c$ and $\lambda = \left(10^{12}/135\times 10^9\right)\times 3\times 10^8\times 10^{-16}$ $= 220\mu m$.

$t_E = \gamma\times 10$ years: $\gamma = (1-u^2/c^2)^{-1/2} = 1.1 \rightarrow t_E = 11$ years.

6. The light source is at rest in frame O where the wavelength is $\lambda = 650nm$ (red). In the car driver's frame, O', the wavelength is $\lambda' = 530nm$ (green). Using the Doppler formula

$$\frac{\lambda}{\lambda'} = \left(\frac{1+u/c}{1-u/c}\right)^{1/2} : \rightarrow \quad \frac{u}{c} = \frac{\lambda^2-\lambda'^2}{\lambda^2+\lambda'^2}$$

substituting the values for λ, λ' we get $u = 6\times 10^7\,ms^{-1}$ or $u = 1.4\times 10^8\,mph$ and the fine $\approx$ £1.4×10^9 or equivalent to winning the national lottery for about 30 weeks!

7. $$1GW = 10^9\,Js^{-1} = 3.1\times 10^{16}\,J/yr = \frac{3.1\times 10^{16}}{1.6\times 10^{19}}\,eV/yr$$

i.e. $\dfrac{3.1\times 10^{16}}{1.6\times 10^{19}\times 200\times 10^6}$ fissions $\approx 10^{27}$ nuclei per year.

$N_A = 6\times 10^{26}$ is the number of ^{235}U nuclei in 235 kg.

Thus mass of ^{235}U required $= \dfrac{235\times 10^{27}}{6\times 10^{26}}$ kg ≈ 400 kg.

Since only 33% efficient, the fuel requirement
≈ 1.2 tonnes/year

1 kg of sea-water (H_2O) contains $\dfrac{1}{3250} \times \dfrac{2}{16+2}$ kg of deuterium, so the number of deuterons/kg of sea-water is $\dfrac{6 \times 10^{26}}{2} \times \dfrac{1}{3250} \times \dfrac{1}{9} \approx 10^{22}$

2 deuterons yield 4 MeV therefore energy content $= 10^{22} \times 2 \times 10^{6} \times 1.6 \times 10^{-19}$
$\approx 3.2 GJ/kg$ of sea water