K. Long, 7 December 2004

Relativity – Lecture 5

Lorentz transformations – Applications I

Lecture 5: LT – Applications I

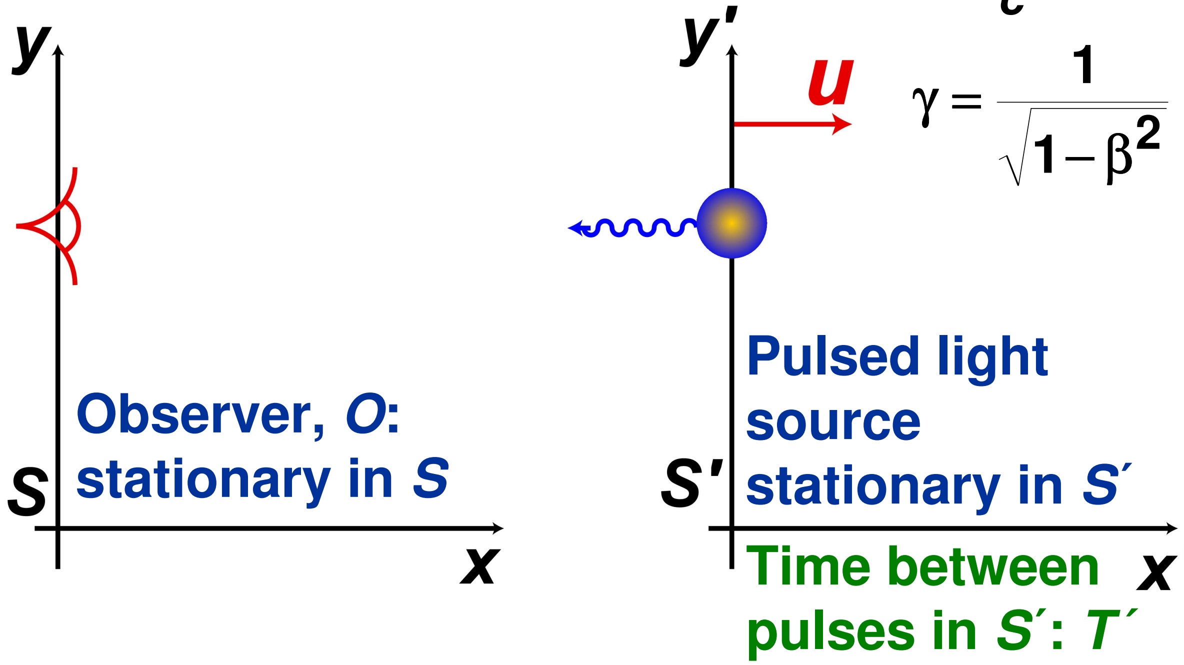
5.1 Relativistic Doppler Effect

- Application:
  - Measure distance to star/galaxy by exploiting 'red shift'
    - Universe expanding
      - Distance to star proportional to speed at which star is receding
      - Pattern of spectral lines shifted by Doppler effect
- Set up problem:
  - Consider pulsed light source, stationary in S'
  - Consider observer, stationary in S
  - Analyse space-time coordinates of two light pulses leaving source and arriving at observer

5.1 Relativistic Doppler Effect

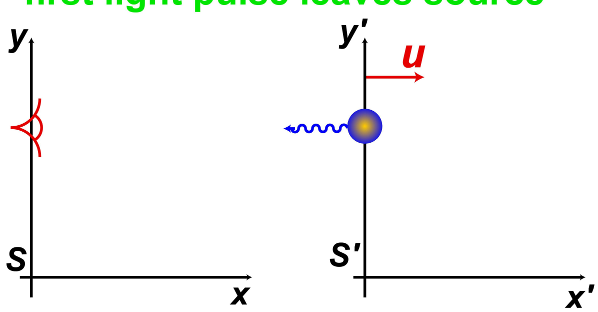
$$\beta = \frac{u}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

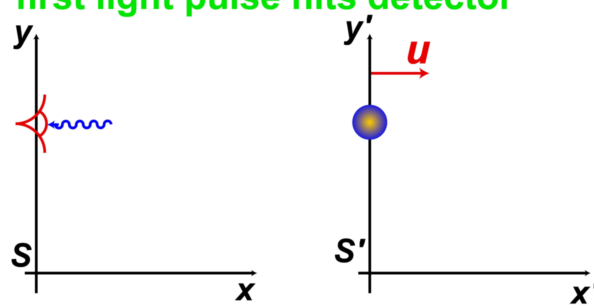


5.1 Relativistic Doppler Effect

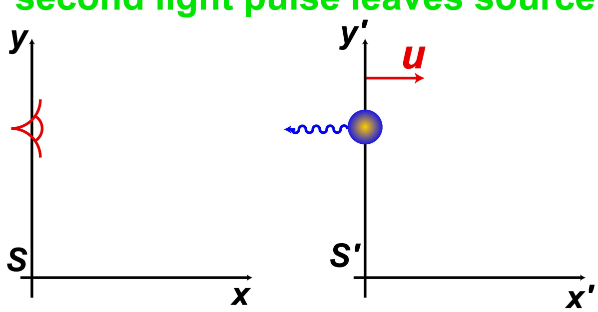
Event 1:
first light pulse leaves source



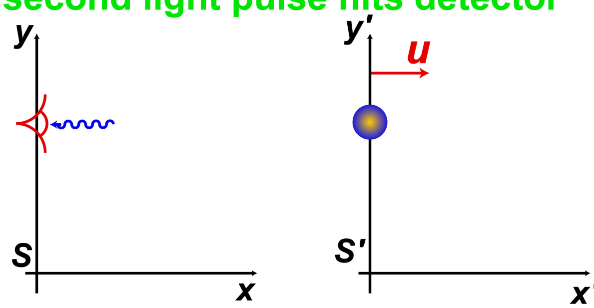
Event 2:
first light pulse hits detector



Event 3:
second light pulse leaves source



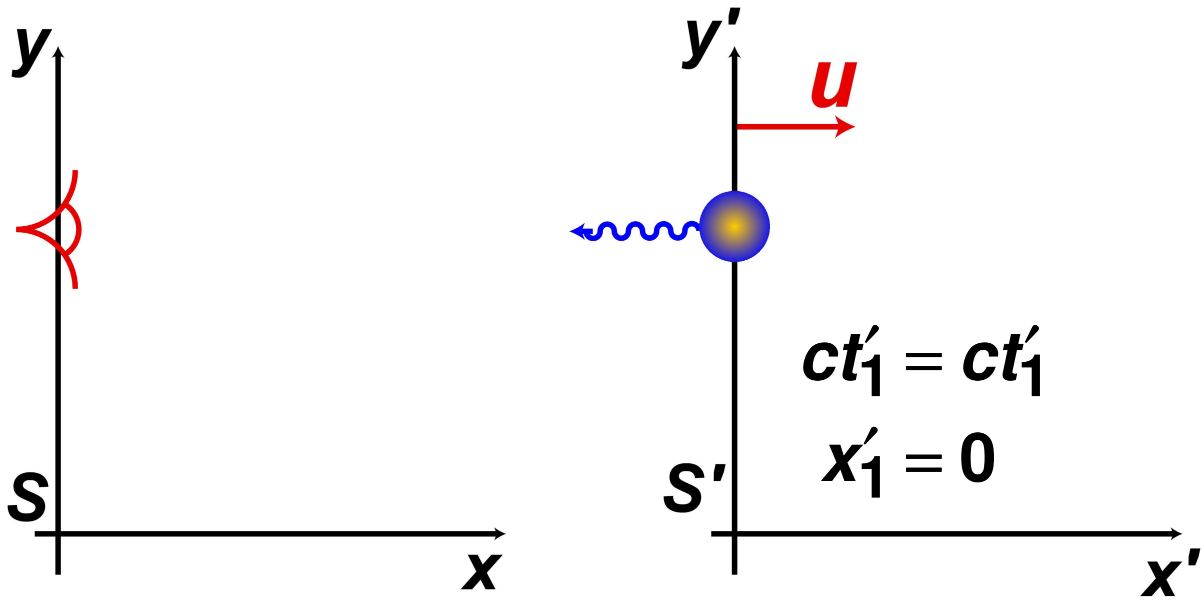
Event 4:
second light pulse hits detector



5.1 Relativistic Doppler effect

Event 1:

first light pulse leaves source



$$ct'_1 = ct'_1$$
$$x'_1 = 0$$

Lect. 4: The Lorentz transformat^n

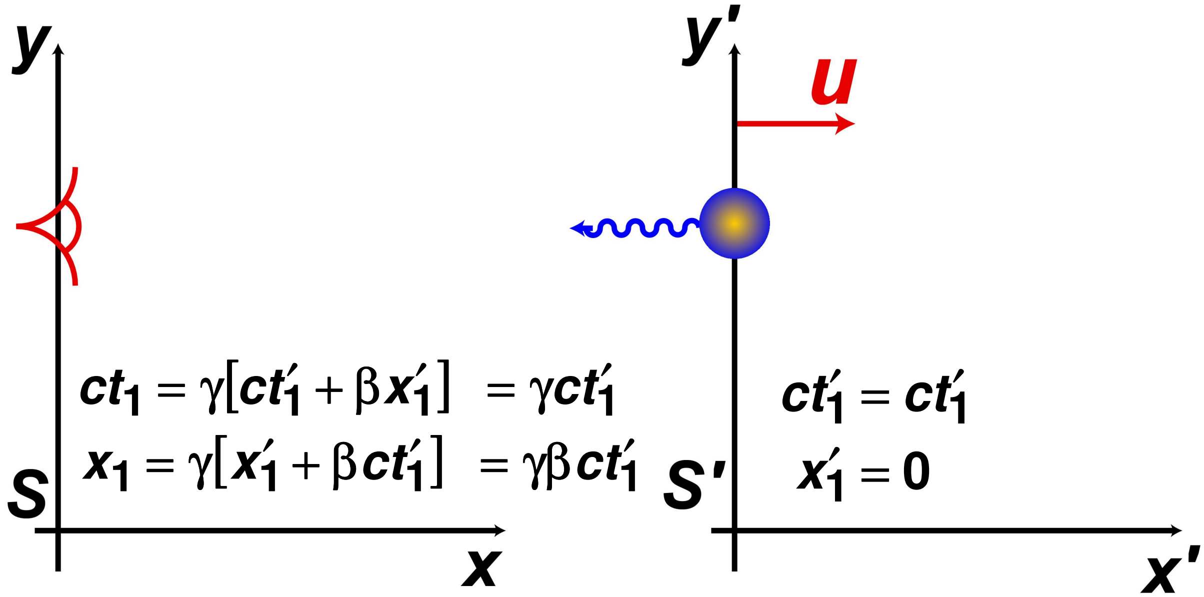
4.7 The Lorentz transformation

Transformation	Inverse transformation
$ct' = \gamma(ct - \beta x)$	$ct = \gamma(ct' + \beta x')$
$x' = \gamma(x - \beta ct)$	$x = \gamma(x' + \beta ct')$
$y' = y$	$y = y'$
$z' = z$	$z = z'$

5.1 Relativistic Doppler effect

Event 1:

first light pulse leaves source



$$ct_1 = \gamma[ct'_1 + \beta x'_1] = \gamma ct'_1$$
$$x_1 = \gamma[x'_1 + \beta ct'_1] = \gamma\beta ct'_1$$

$$ct'_1 = ct'_1$$
$$x'_1 = 0$$

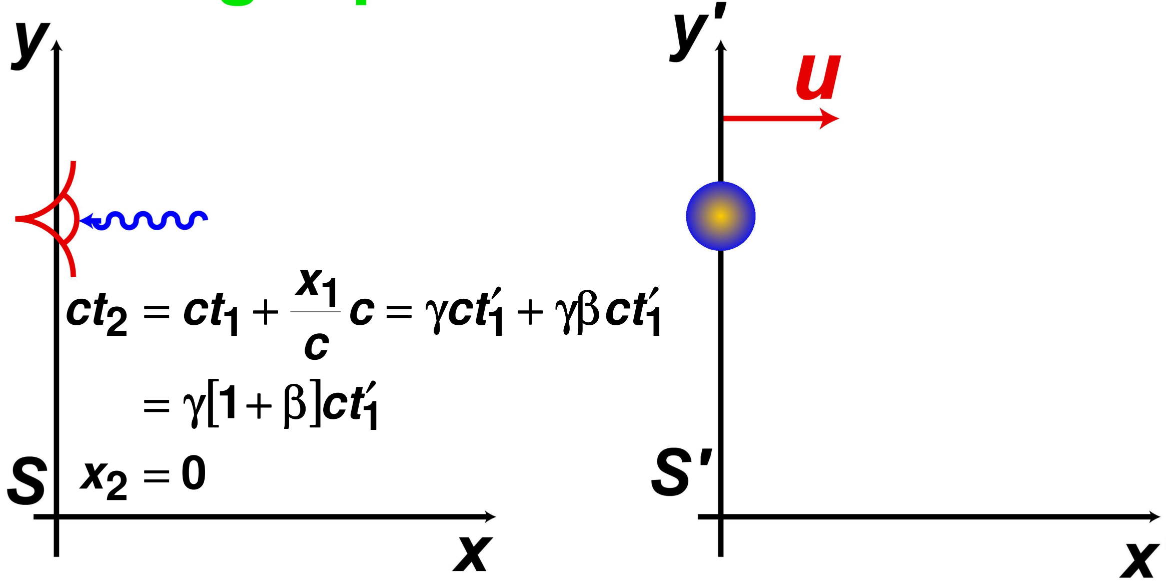
5.1 Relativistic Doppler effect

Event	#	In S	In S'
First pulse leaves	1	$ct_1 = \gamma[ct'_1 + \beta x'_1] = \gamma ct'_1$ $x_1 = \gamma[x'_1 + \beta ct'] = \gamma\beta ct'_1$	ct'_1 $x'_1 = 0$
First pulse arrives	2		
Second pulse leaves	3		
Second pulse arrives	4		

5.1 Relativistic Doppler effect

Event 2:

first light pulse hits detector



$$ct_2 = ct_1 + \frac{x_1}{c}c = \gamma ct_1' + \gamma\beta ct_1'$$

$$= \gamma[1+\beta]ct_1'$$

$$x_2 = 0$$

Lect. 4: The Lorentz transformat^n

4.7 The Lorentz transformation

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$ct' = \gamma(ct - \beta x)$	$ct = \gamma(ct' + \beta x')$
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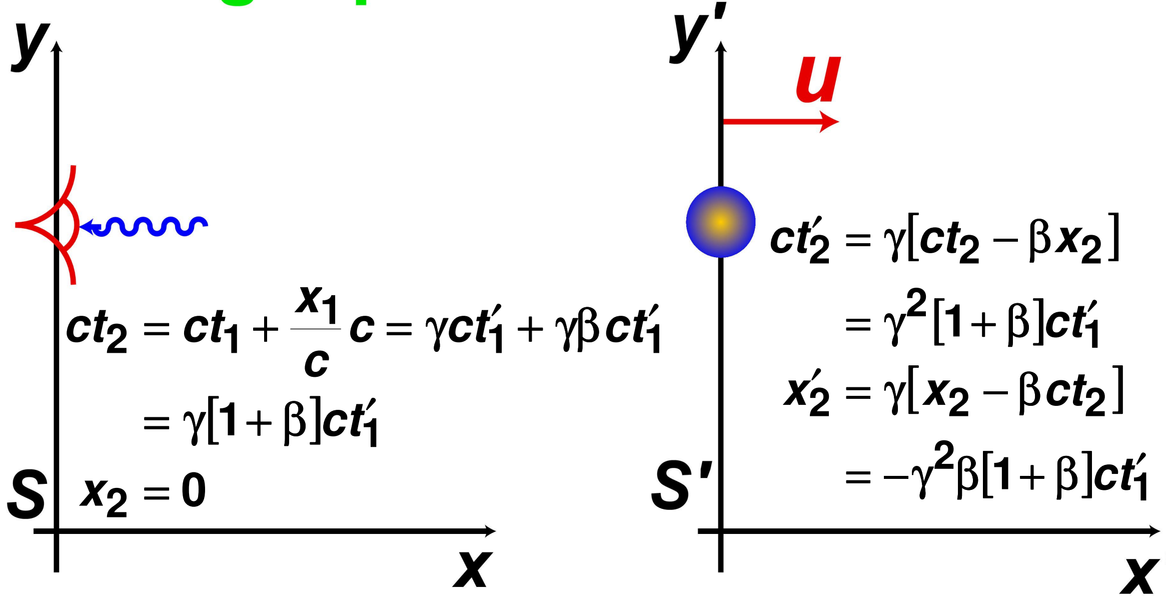
5.1 Relativistic Doppler effect

Event 2:
first light pulse hits detector



$$ct_2 = ct_1 + \frac{x_1}{c}c = \gamma ct_1' + \gamma\beta ct_1'$$
$$= \gamma[1+\beta]ct_1'$$
$$x_2 = 0$$

$$ct_2' = \gamma[ct_2 - \beta x_2]$$
$$= \gamma^2[1+\beta]ct_1'$$
$$x_2' = \gamma[x_2 - \beta ct_2]$$
$$= -\gamma^2\beta[1+\beta]ct_1'$$

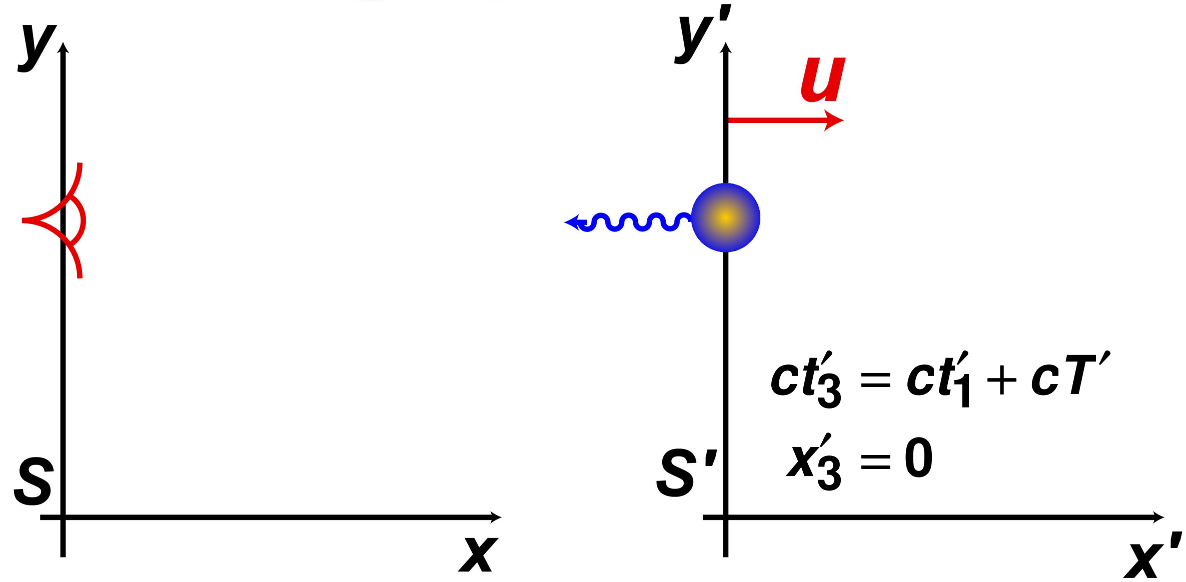
5.1 Relativistic Doppler effect

Event	#	In S	In S'
First pulse leaves	1	$ct_1 = \gamma[ct_1' + \beta x_1'] = \gamma ct_1'$ $x_1 = \gamma[x_1' + \beta ct'] = \gamma\beta ct_1'$	ct_1' $x_1' = 0$
First pulse arrives	2	$ct_2 = ct_1 + \frac{x_1}{c}c = ct_1 + \gamma\beta ct_1'$ $= \gamma ct_1' + \gamma\beta ct_1' = \gamma[1+\beta]ct_1'$ $x_2 = 0$	$ct_2' = \gamma[ct_2 - \beta x_2] = \gamma^2[1+\beta]ct_1'$ $x_2' = \gamma[x_2 - \beta ct_2] = -\gamma^2\beta[1+\beta]ct_1'$
Second pulse leaves	3		
Second pulse arrives	4		

5.1 Relativistic Doppler Effect

Event 3: T' is period in S'

second light pulse leaves source



$$ct'_3 = ct'_1 + cT'$$
$$x'_3 = 0$$

Lect. 4: The Lorentz transformat^{n}

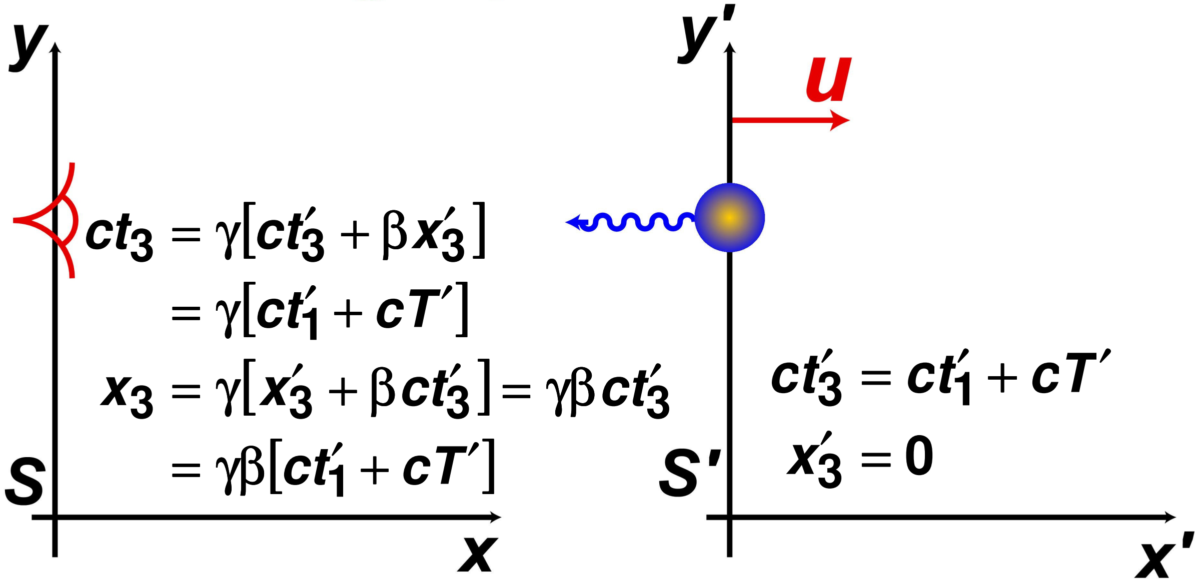
4.7 The Lorentz transformation

Transformation	Inverse transformation
$ct' = \gamma(ct - \beta x)$	$ct = \gamma(ct' + \beta x')$
$x' = \gamma(x - \beta ct)$	$x = \gamma(x' + \beta ct')$
$y' = y$	$y = y'$
$z' = z$	$z = z'$

5.1 Relativistic Doppler Effect

Event 3:

second light pulse leaves source



$$ct_3 = \gamma[ct'_3 + \beta x'_3] = \gamma[ct'_1 + cT']$$

$$x_3 = \gamma[x'_3 + \beta ct'_3] = \gamma\beta ct'_3 = \gamma\beta[ct'_1 + cT']$$

$$ct'_3 = ct'_1 + cT'$$

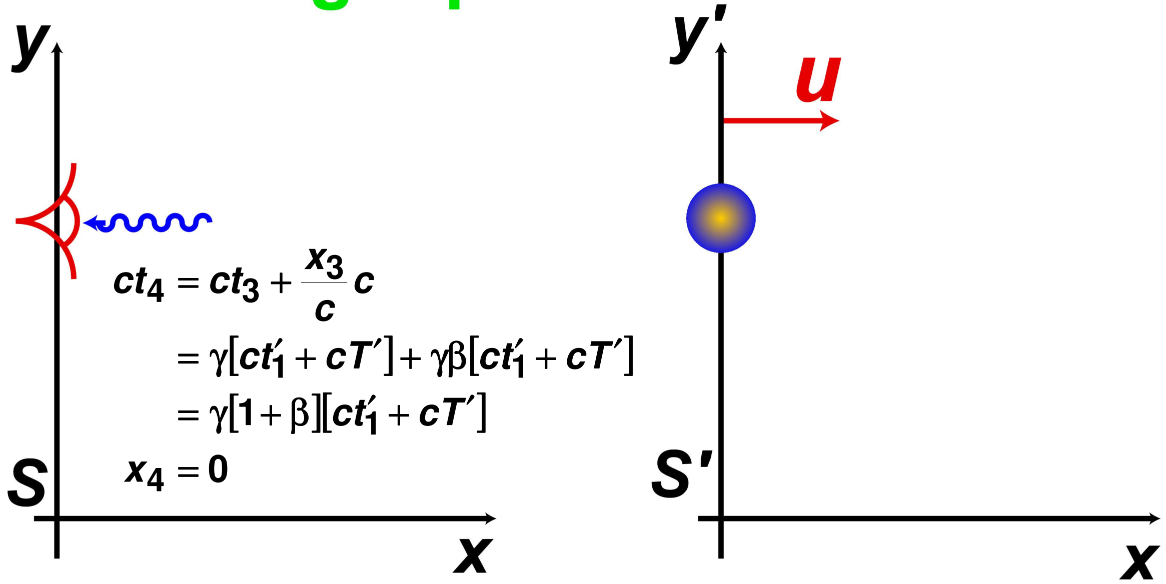
$$x'_3 = 0$$

5.1 Relativistic Doppler Effect

Event	#	In S	In S'
First pulse leaves	1	$ct_1 = \gamma[ct'_1 + \beta x'_1] = \gamma ct'_1$ $x_1 = \gamma[x'_1 + \beta ct'] = \gamma\beta ct'_1$	ct'_1 $x'_1 = 0$
First pulse arrives	2	$ct_2 = ct_1 + \frac{x_1}{c}c = ct_1 + \gamma\beta ct'_1$ $= \gamma ct'_1 + \gamma\beta ct'_1 = \gamma[1+\beta]ct'_1$ $x_2 = 0$	$ct'_2 = \gamma[ct_2 - \beta x_2] = \gamma^2[1+\beta]ct'_1$ $x'_2 = \gamma[x_2 - \beta ct_2] = -\gamma^2\beta[1+\beta]ct'_1$
Second pulse leaves	3	$ct_3 = \gamma[ct'_3 + \beta x'_3] = \gamma[ct'_1 + cT']$ $x_3 = \gamma[x'_3 + \beta ct'_3] = \gamma\beta ct'_3$ $= \gamma\beta[ct'_1 + cT']$	$ct'_3 = ct'_1 + cT'$ $x'_3 = 0$
Second pulse arrives	4		

5.1 Relativistic Doppler Effect

Event 4:
second light pulse hits detector



$$ct_4 = ct_3 + \frac{x_3}{c}c$$
$$= \gamma[ct_1' + cT'] + \gamma\beta[ct_1' + cT']$$
$$= \gamma[1+\beta][ct_1' + cT']$$
$$x_4 = 0$$

Lect. 4: The Lorentz transformat^{n}

4.7 The Lorentz transformation

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$ct' = \gamma(ct - \beta x)$	$ct = \gamma(ct' + \beta x')$
$x' = \gamma(x - \beta ct)$	$x = \gamma(x' + \beta ct')$
$y' = y$	$y = y'$
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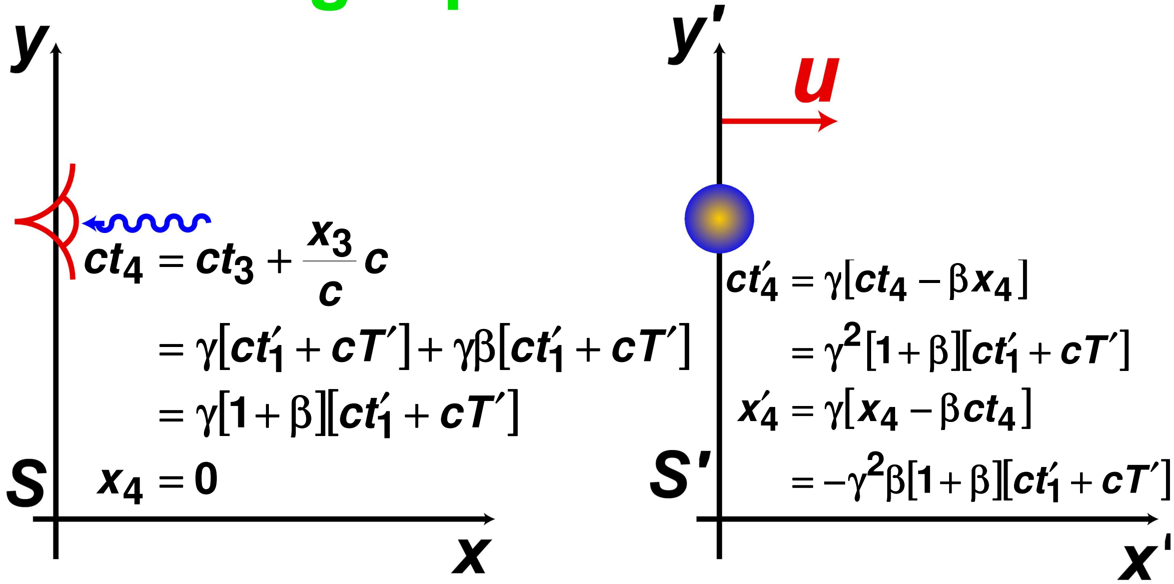
5.1 Relativistic Doppler Effect

Event 4:

second light pulse hits detector



In S:

$$ct_4 = ct_3 + \frac{x_3}{c}c$$
$$= \gamma[ct_1' + cT'] + \gamma\beta[ct_1' + cT']$$
$$= \gamma[1+\beta][ct_1' + cT']$$
$$x_4 = 0$$

In S':

$$ct_4' = \gamma[ct_4 - \beta x_4]$$
$$= \gamma^2[1+\beta][ct_1' + cT']$$
$$x_4' = \gamma[x_4 - \beta ct_4]$$
$$= -\gamma^2\beta[1+\beta][ct_1' + cT']$$

5.1 Relativistic Doppler Effect

Event	#	In S	In S'
First pulse leaves	1	$ct_1 = \gamma[ct_1' + \beta x_1'] = \gamma ct_1'$ $x_1 = \gamma[x_1' + \beta ct'] = \gamma\beta ct_1'$	ct_1' $x_1' = 0$
First pulse arrives	2	$ct_2 = ct_1 + \frac{x_1}{c}c = ct_1 + \gamma\beta ct_1'$ $= \gamma ct_1' + \gamma\beta ct_1' = \gamma[1+\beta]ct_1'$ $x_2 = 0$	$ct_2' = \gamma[ct_2 - \beta x_2] = \gamma^2[1+\beta]ct_1'$ $x_2' = \gamma[x_2 - \beta ct_2] = -\gamma^2\beta[1+\beta]ct_1'$
Second pulse leaves	3	$ct_3 = \gamma[ct_3' + \beta x_3'] = \gamma[ct_1' + cT']$ $x_3 = \gamma[x_3' + \beta ct_3'] = \gamma\beta ct_3'$ $= \gamma\beta[ct_1' + cT']$	$ct_3' = ct_1' + cT'$ $x_3' = 0$
Second pulse arrives	4	$ct_4 = ct_3 + \frac{x_3}{c}c$ $= \gamma[ct_1' + cT'] + \gamma\beta[ct_1' + cT']$ $= \gamma[1+\beta][ct_1' + cT']$ $x_4 = 0$	$ct_4' = \gamma[ct_4 - \beta x_4]$ $= \gamma^2[1+\beta][ct_1' + cT']$ $x_4' = \gamma[x_4 - \beta ct_4]$ $= -\gamma^2\beta[1+\beta][ct_1' + cT']$

5.1 Relativistic Doppler Effect

- **Calculate period in *S*:**

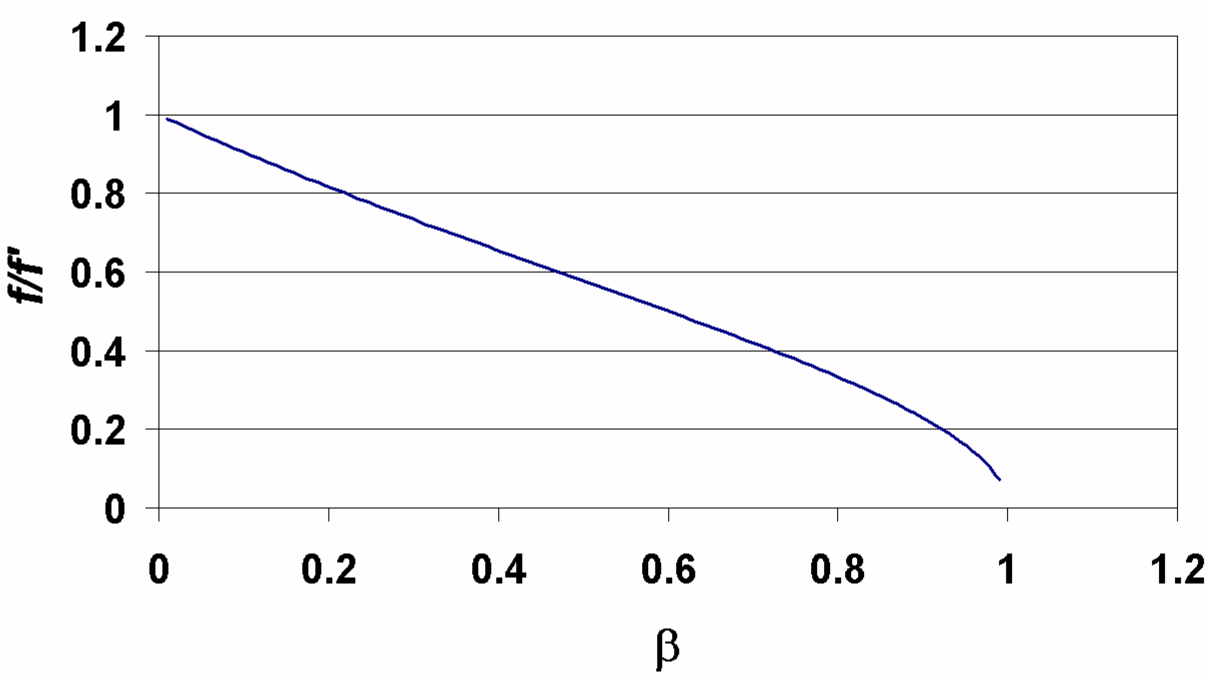
$$
\begin{aligned}
cT &= ct_4 - ct_2 \\
&= \gamma[1+\beta](ct_1' + cT') - \gamma[1+\beta]ct_1' \\
&= \gamma[1+\beta]cT' = \frac{1+\beta}{\sqrt{(1+\beta)(1-\beta)}}cT' \\
&= cT'\sqrt{\frac{1+\beta}{1-\beta}}
\end{aligned}
$$

- **Convert to frequency:**

$$
\boxed{f = f'\sqrt{\frac{1-\beta}{1+\beta}}}
$$

5.1 Relativistic Doppler Effect

Red shift

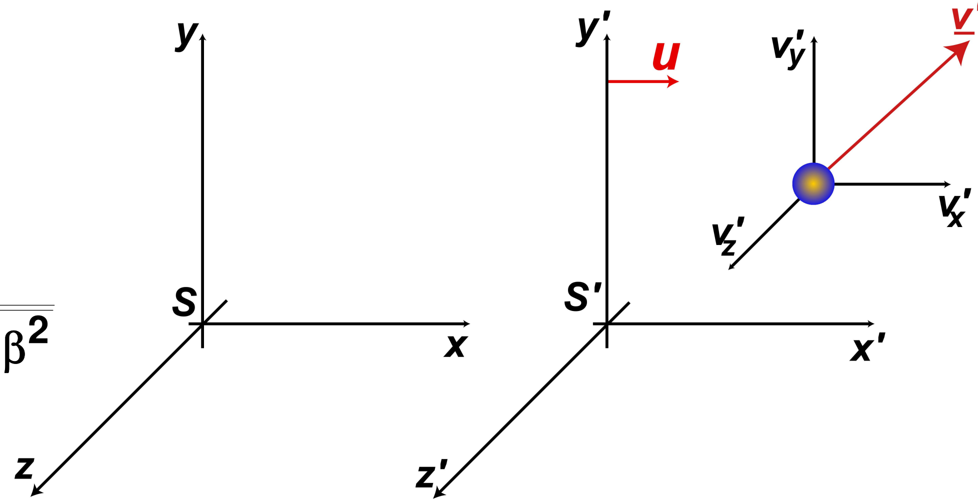


Lecture 5: LT – Applications I

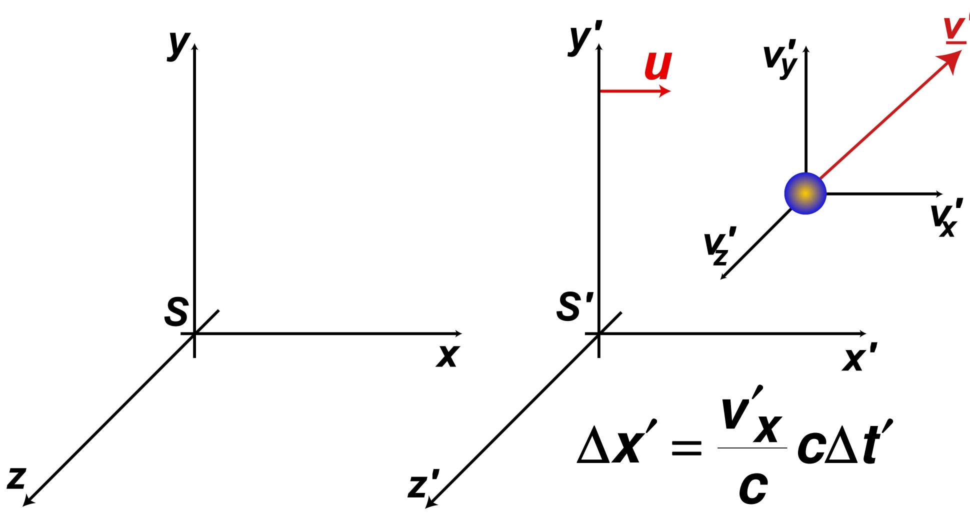
5.2 Transformation of velocity

$$\beta = \frac{u}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$



5.2 Transformation of velocity



Seen from *S´*

$$\Delta x' = \frac{v'_x}{c} c\Delta t'$$

$$\Delta y' = \frac{v'_y}{c} c\Delta t'$$

$$\Delta z' = \frac{v'_z}{c} c\Delta t'$$

Lect. 4: The Lorentz transformat^{n}

4.7 The Lorentz transformation

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$x' = \gamma(x - \beta ct)$	$x = \gamma(x' + \beta ct')$
$y' = y$	$y = y'$
$z' = z$	$z = z'$

5.2 Transformation of velocity

- Use Lorentz transformation to obtain space and time intervals as seen in *S*:

$$c\Delta t = \gamma[c\Delta t' + \beta\Delta x']$$

$$\Delta x = \gamma[\Delta x' + \beta c\Delta t']$$

$$\Delta y = \Delta y'$$

$$\Delta z = \Delta z'$$

- Divide Δx, Δy, Δz by $c\Delta t$ to obtain velocity as observed in *S*

5.2 Transformation of velocity

$$\frac{v_x}{c} = \frac{\Delta x}{c\Delta t} = \frac{\gamma[\Delta x' + \beta(c\Delta t')]}{\gamma[c\Delta t' + \beta(\Delta x')]} = \frac{\frac{v'_x}{c} + \beta}{1 + \beta\frac{v'_x}{c}}$$

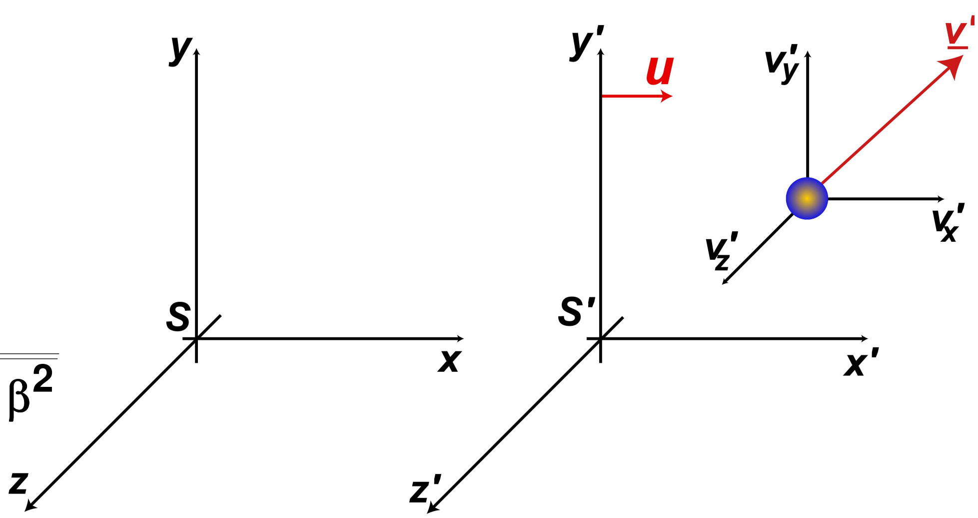
$$\frac{v_y}{c} = \frac{\Delta y}{c\Delta t} = \frac{\Delta y'}{\gamma[c\Delta t' + \beta(\Delta x')]} = \frac{1}{\gamma}\left[\frac{\frac{v'_y}{c}}{1 + \beta\frac{v'_x}{c}}\right]$$

$$\frac{v_z}{c} = \frac{\Delta z}{c\Delta t} = \frac{\Delta z'}{\gamma[c\Delta t' + \beta(\Delta x')]} = \frac{1}{\gamma}\left[\frac{\frac{v'_z}{c}}{1 + \beta\frac{v'_x}{c}}\right]$$

5.2 Transformation of velocity

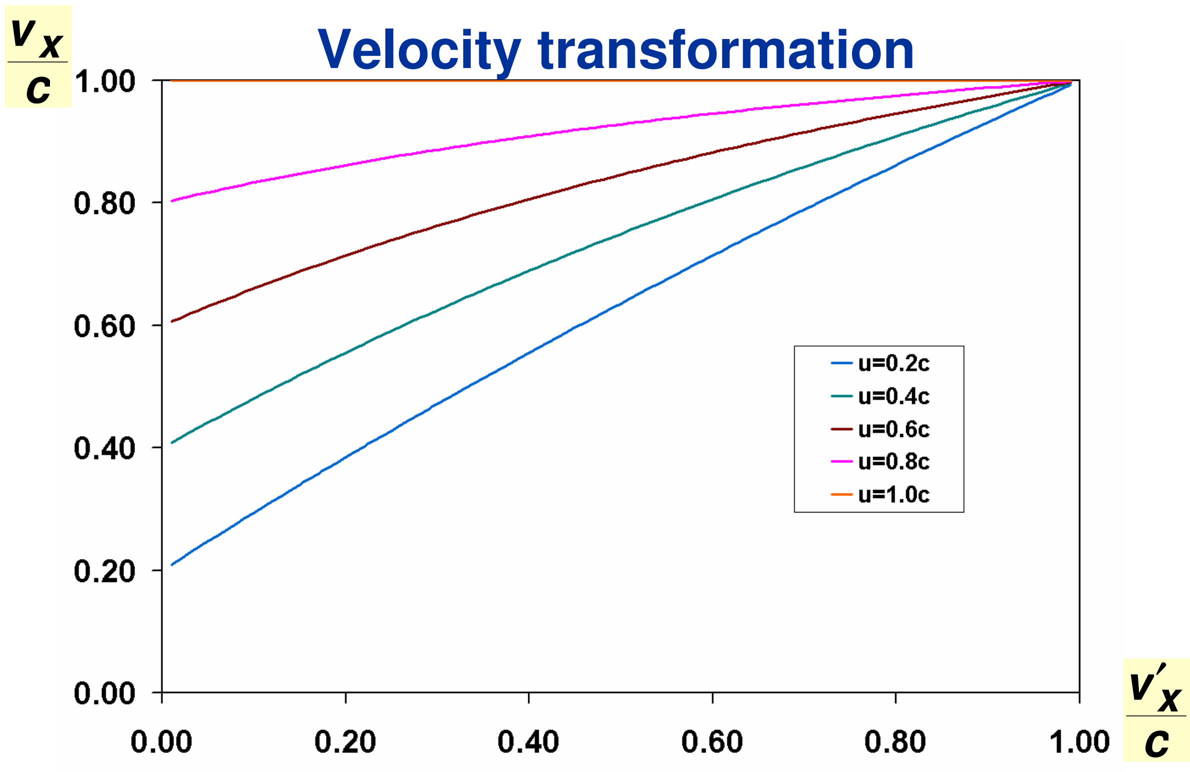
$$\beta = \frac{u}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$



$\frac{v_x}{c} = \frac{\frac{v'_x}{c} + \beta}{1 + \beta\frac{v'_x}{c}}$	$\frac{v_y}{c} = \frac{1}{\gamma}\left[\frac{\frac{v'_y}{c}}{1 + \beta\frac{v'_x}{c}}\right]$	$\frac{v_z}{c} = \frac{1}{\gamma}\left[\frac{\frac{v'_z}{c}}{1 + \beta\frac{v'_x}{c}}\right]$

5.2 Transformation of velocity



Lecture 5: LT – Applications I

5.3 Correspondence principle

- **Low velocity limit of relativistic transformation must yield Galilean transformation**
  - Galilean transformation of velocities requires that $u << c$, $v << c$ and $v' << c$
  - Under these conditions, Galilean transformation of velocity:

$$v_x = v'_x + u \quad v_y = v'_y \quad v_z = v'_z$$

  - is recovered.

5.3 Correspondence principle

- For example:

$$\frac{v_x}{c} = \frac{\frac{v'_x}{c} + \beta}{1 + \beta \frac{v'_x}{c}}$$

$$v_x = \frac{v'_x + u}{1 + \frac{u v'_x}{c^2}}$$

$$u << c \text{ and } v'_x << c$$

$$v_x \approx v'_x + u$$