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Relativity – key notes

1. Definitions

Consider two inertial frames S and S'. S' moves at a constant velocity relative to S. The relative velocity of S' with respect to S is u parallel to the x axis in S. The coordinate axes of S and S' coincide at the instant $t = t' = 0$. The relative motion of the two frames is shown schematically in figure 1.

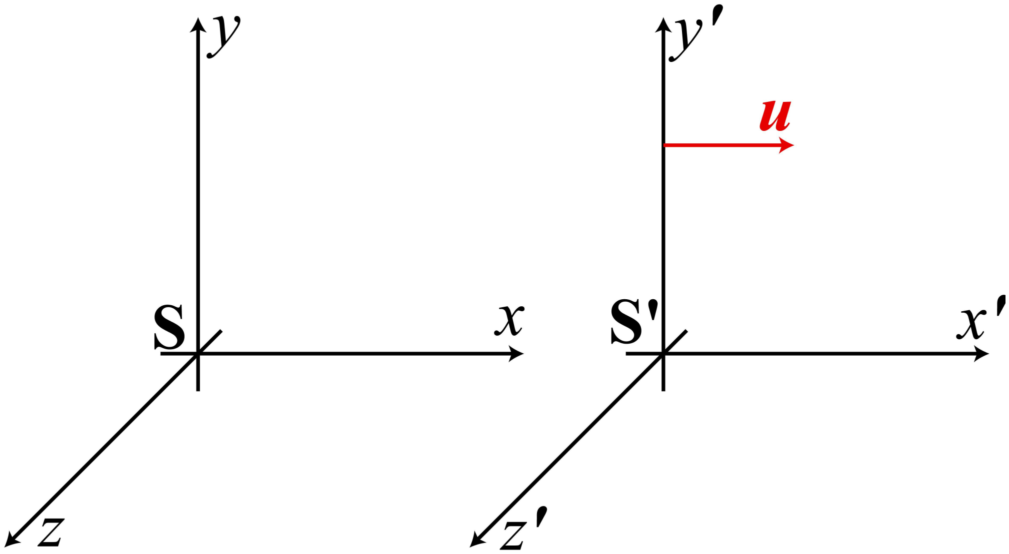

Figure 1: Inertial frames S and S'. S' moves at a constant velocity u *parallel to the* x *axis in S. The coordinate axes of the two frames coincide when $t = t' = 0$.*

The relative velocity in units of the speed of light, β, is defined to be

$$\beta = \frac{u}{c} \tag{1}$$

where c is the speed of light. The variable γ is defined by

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2}$$

An invariant quantity has the same *numerical value* in all inertial frames. The speed of light is an example of an invariant quantity. The speed of light is the same for all inertial observers.

A covariant expression takes the same *form* in all inertial frames. The Lorentz transformation equations are examples of covariant expressions.

The space-time coordinates of an event are conveniently expressed in terms of the space coordinates x, y, z in S and x', y', z' in S'. It is convenient, though not essential, to measure the time coordinate of an event by ct in S and ct' in S'. Since the speed of light is invariant this has no effect other than to make length the dimension in which time is measured.

The principles of relativity are that:

- The laws of physics are covariant;
- The speed of light is invariant.

2. Lorentz transformation

The Lorentz transformation of space and time between two inertial frames S and S' and the inverse transformation between S' and S may be written:

$$
\begin{array}{ll}
\text{Transformation : } S \to S' & \text{Inverse transformation : } S' \to S \\
ct' = \gamma(ct - \beta x) & ct = \gamma(ct' + \beta x') \\
x' = \gamma(x - \beta ct) & x = \gamma(x' + \beta ct') \\
y' = y & y = y' \\
z' = z & z = z'
\end{array}
\tag{3}
$$

3. Time dilation

Consider two events that take place at the same point in space in S'. The space-time coordinates of these two events may be written:

$$
\begin{array}{cc}
\text{Event 1} & \text{Event 2} \\
x'_1, y'_1, z'_1 & x'_2, y'_2, z'_2 \\
ct'_1 & ct'_2
\end{array}
\tag{4}
$$

The separation of the two events in S' is given by:

$$\Delta x' = \Delta y' = \Delta z' = 0 \tag{5}$$

Application of the inverse Lorentz transformation (equation 3) allows the time interval measured in S to be expressed in terms of the time interval expressed in S' as follows:

$$c\Delta t = \gamma c\Delta t' \tag{6}$$

This is the 'time-dilation' formula.

4. Length contraction

Consider two events that take place at the same instant in S. The space-time coordinates of these two events may be written:

$$
\begin{array}{cc}
\text{Event 3} & \text{Event 4} \\
x_3, y_3, z_3 & x_4, y_4, z_4 \\
ct_3 & ct_4
\end{array}
\tag{7}
$$

The time interval between event 3 and event 4 measured in S is $c\Delta t = 0$. The Lorentz transformation (equation 3) allows the separation of the two events measured in S' to be related to that measured in S. The Lorentz transformation relating the x and x' coordinates can be used to relate the separation of the two events in x to the separation of the two events in x' as follows:

$$\Delta x = \frac{\Delta x'}{\gamma} \tag{8}$$

This is the 'length-contraction' formula.

5. Simultaneity and some more definitions

Two events can only be simultaneous in all inertial frames if the two events take place at the same point in space.

The proper time is defined to be the time interval between two events measured in the inertial frame in which the two events take place at the same point in space. This frame is also referred to as the rest frame.

The proper length of an object is defined to be the length of an object measured in the frame in which the object is at rest, i.e. in the rest frame of the object.

6. The invariant interval

Consider two events that take place at the same point in an inertial frame S''. Further, consider the frame S'' to be in uniform relative motion with respect to S and S'. Let the time interval between the two events measured in S'' (the proper time) be τ. The proper time can be determined by measuring the separation and the time interval between the two events in another inertial frame and evaluating:

$$(c\tau)^2 = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = (c\Delta t')^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2 . \tag{9}$$

The invariant interval is defined as follows:

$$\text{Invariant interval} = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 . \tag{10}$$

7. Energy and momentum

Consider a particle of mass m moving at a speed v in S. The speed of the particle measured in units of the speed of light is

$$\beta_m = \frac{v}{c} . \tag{11}$$

It is once again convenient to define the variable γ_m by

$$\gamma_m = \frac{1}{\sqrt{1-\beta_m^2}} . \tag{12}$$

The relativistic expression for the energy of the particle is

$$E = \gamma_m mc^2 . \tag{13}$$

The relativistic expression for the momentum of the particle is

$$cp = \gamma_m \beta_m mc^2 . \tag{14}$$

Note that the momentum has been multiplied by c for convenience.

The principle of equivalence requires that the non-relativistic expressions for energy and momentum are recovered in the low velocity limit. The low velocity limit $\beta_m << 1$ is obtained by taking the binomial expansion of the expression for γ_m (equation 12) and keeping only the first term in β_m as follows

$$\gamma_m = \frac{1}{\sqrt{1-\beta_m^2}} = \left(1-\beta_m^2\right)^{-\frac{1}{2}} \approx 1 + \tfrac{1}{2}\beta_m^2 . \tag{15}$$

This expression for γ_m may be substituted in equations 13 and 14 and, after discarding non-leading terms, yields:

$$\begin{aligned} E &\approx mc^2 + \tfrac{1}{2}mv^2 \\ p &\approx mv \end{aligned} \tag{16}$$

The non-relativistic expression for momentum is recovered. The expression for the energy contains two terms. The term mc^2 is referred to as the rest-mass energy. The non-relativistic kinetic energy $(1/2)mv^2$ appears in the second term.

8. The Lorentz transformation of energy and momentum

The energy and momentum of a particle of mass m measured in the frame S are E, p_x, p_y, p_z. The energy and momentum of the same particle measured in the frame S' are E', p'_x, p'_y, p'_z. The Lorentz transformation and its inverse may be written:

$$\begin{array}{ll} \text{Transformation: } \mathrm{S} \rightarrow \mathrm{S}' & \text{Inverse transformation: } \mathrm{S}' \rightarrow \mathrm{S} \\ E' = \gamma(E - \beta c p_x) & E = \gamma(E' + \beta c p'_x) \\ c p'_x = \gamma(c p_x - \beta E) & c p_x = \gamma(c p'_x + \beta E') \\ c p'_y = c p_y & c p_y = c p'_y \\ c p'_z = c p_z & c p_z = c p'_z \end{array} \qquad (17)$$

9. Energy and momentum conservation

Consider a system of n particles. The total energy E_{Tot} and the total momentum $\mathbf{P}_{\mathrm{Tot}}$ of the system of particles is given by

$$\begin{aligned} E_{\mathrm{Tot}} &= \sum_{i=1}^{n} E_i \\ \mathbf{P}_{\mathrm{Tot}} &= \sum_{i=1}^{n} \mathbf{p}_i \end{aligned} \qquad (18)$$

where E_i and $\mathbf{p}_i$ are the energy and momentum of the i^{th} particle.

In a decay process or in a relativistic collision the total energy and the total momentum are conserved. This has the consequence that rest mass can be changed in to kinetic energy, for example in the decay of a radioactive nucleus. Alternatively, kinetic energy can be changed into rest mass, for example in the creation of a heavy particle in the energetic collision of two particles.

An elastic collision is one in which the particles that enter in the initial state are the same as those that appear in the final state. An inelastic process is one in which particles are created or destroyed.