November 22, 2007

Relativity Problem Sheet 1

1.1 What's conserved?

Two inertial frames of reference are in uniform relative motion. Which of the following quantities are necessarily measured to be the same in both frames?

1. The speed of light in vacuum
2. The speed of an electron
3. The charge of an electron
4. The kinetic energy of an electron
5. The electric field at a given point
6. The order of elements in the periodic table
7. The time interval between two events
8. Newton's First Law

1.2 Alpha Centauri

Alpha Centauri is 4.4 light years away (as measured in the Earth frame of reference). If astronauts wanted to reach it in ten years as reckoned by their own clocks, at what (constant) speed would they need to travel ? How long would the trip take according to Earth time ?

1.3 Some Numbers

(a) At what speed is a metre rule moving relative to an observer who finds its length only $750\,\mathrm{mm}$?

(b) According to Mr Spock's clock, Captain Kirk's clock is running slow by 5%. What is their relative speed ?

(c) Plot a graph of the Lorentz factor, $\gamma = 1/\sqrt{1-u^2/c^2}$, as a function of u/c from 0 to 1 in steps of 0.1. Above what speed are relativistic corrections to Galilean kinematics more than 10% ?

(d) Use the binomial expansion to find an approximation for γ when $u \ll c$, up to terms of order u^2/c^2.

(e) An observer O' moves along the positive x-axis with velocity $u = 0.9c$ relative to an observer O. The space-time origins of O and O' coincide. Use Lorentz transformations to answer the following:

   (i) What are the space-time coordinates in O' of events A and B occurring with coordinates $(x_A, t_A) = (0, 100\,\mathrm{ns})$; $(x_B, t_B) = (-100\,\mathrm{m}$,0) in O ?

   (ii) What are the space-time coordinates in O of events C and D occurring with coordinates $(x'_C, t'_C) = (0, 100\,\mathrm{ns})$; $(x'_D, t'_D) = (-100\,\mathrm{m}$,0) in O' ?

Answers to numerical exercises

(a) $0.66c$, (b) $0.3c$.
(e) (i)$(-62\,\mathrm{m}, 230\,\mathrm{ns})$, $(-230\,\mathrm{m}, 700\,\mathrm{ns})$, (ii) $(62\,\mathrm{m}, 230\,\mathrm{ns})$, $(-230\,\mathrm{m}, -700\,\mathrm{ns})$.

1.4 Inverse Lorentz Transform

The Lorentz transformation $LT(u)$ equations

$$\begin{aligned} x' &= \gamma(x - ut) \\ y', z' &= y, z \\ t' &= \gamma(t - ux/c^2) \end{aligned}$$

express the space-time coordinates of an event in O', in terms of those for the *same* event in O, where O' moves at velocity u relative to O along the x-axis. Solve for (x, t) in terms of (x', t') and hence show that the inverse transformation $LT^{-1}(u) = LT(-u)$.

1.5 A Spherical Wavefront

A flash of light at the origin O results in a spherical wavefront which propagates outwards from O at speed c. The equation of the wavefront (at time t) is

$$(x^2 + y^2 + z^2) = c^2t^2 \ .$$

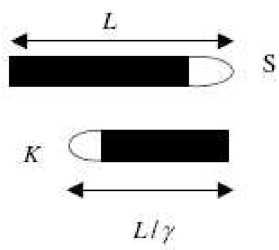
Another observer O', moving at velocity u along the x-axis observes the wavefront.
Show that the wave front is **not** squashed by a Lorentz contraction, but that it remains spherical; i.e. the equation is *covariant*

$$(x'^2 + y'^2 + z'^2) = c^2t'^2.$$

Hence show that the quantity $c^2t^2 - (x^2 + y^2 + z^2)$ is *invariant*.

1.6 Hit or Miss?

This question is harder than the others
Captain Kirk and Mr Spock pilot identical spaceships of length L (in their own respective rest frames). A prearranged practice session involves Kirk and Spock approaching each other, almost head-on with high relative speed, u. At the instant when the *nose* of Spock's ship is level with the *tail* of Kirk's ship then Spock will fire a warning shot from a gun in his tail in front of Kirk (see figure).
Spock has reassured Kirk that this is harmless due to the Lorentz contraction of Kirk's ship to a length L/γ as illustrated. Imagine Kirk's worried expression when approaching head-on and observing Spock's ship contracted!
Ignore the transverse separation of the ships and the bullet travel times. Does the bullet hit or miss ? Analyse (using Lorentz transformations) the problem from both Kirk's and Spock's viewpoint.



Practice session from S's viewpoint