Relativity — Lecture 10

- Summary of Lecture 9
- Big Summary
- The Twins Paradox

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100 years of living science

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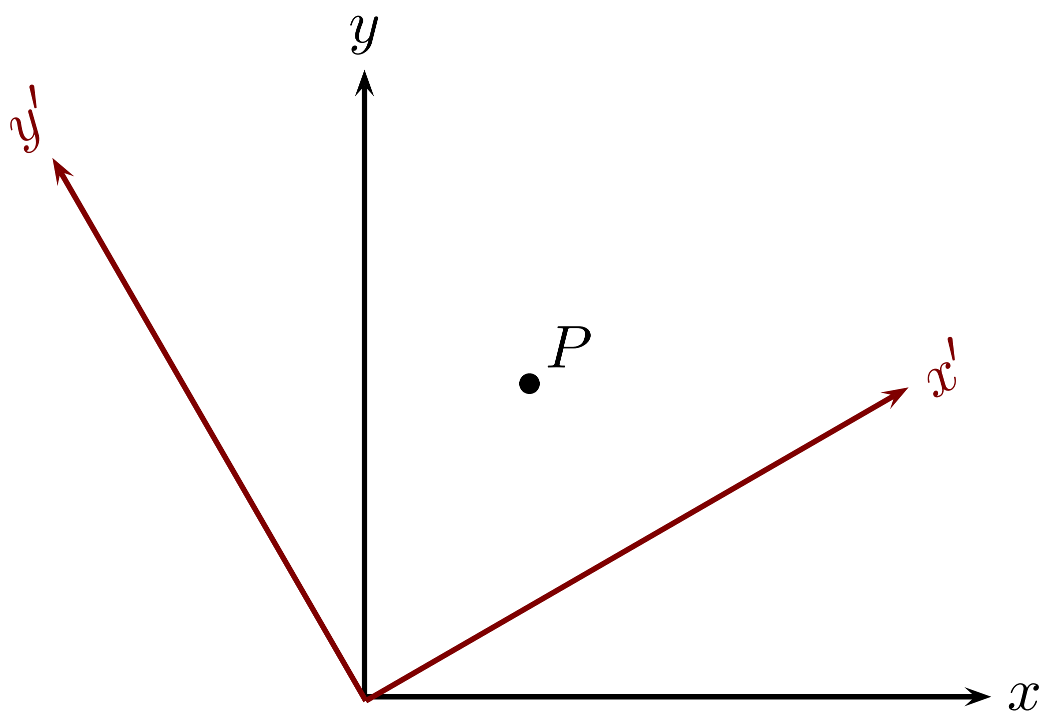
Lecture 9

Revision

Rotations and Lorentz Transforms



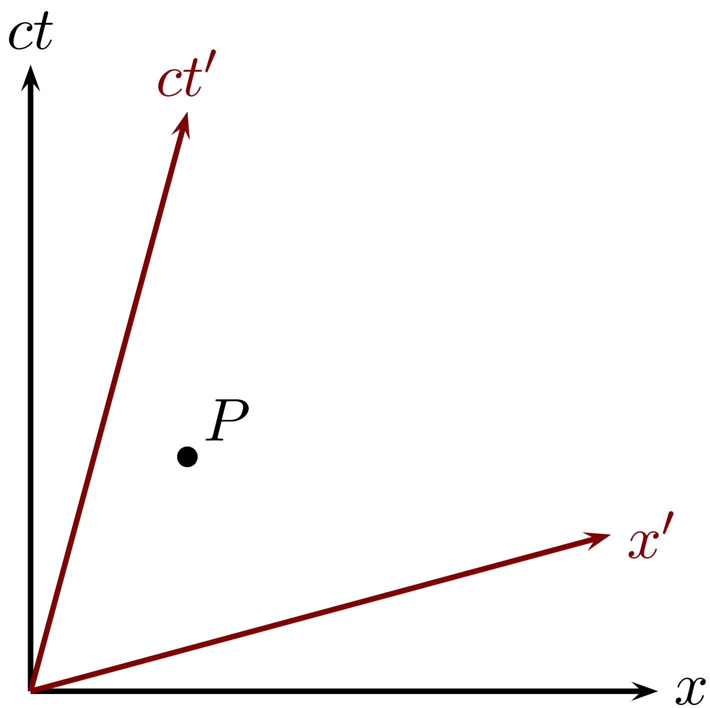
Rotation in 2D:

$$\begin{aligned} x' &= x\cos\alpha + y\sin\alpha \\ y' &= y\cos\alpha - x\sin\alpha \end{aligned}$$

Invariants:

Rotation: $r'^2 = x'^2 + y'^2 + z'^2 \quad = \quad x^2 + y^2 + z^2 = r^2$

Rotations and Lorentz Transforms



Rotation in 2D:

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Lorentz Transform:

$$\begin{aligned} x' &= \gamma\,(x - \beta ct) \\ ct' &= \gamma\,(ct - \beta x) \end{aligned}$$

Invariants:

Rotation: $r'^2 = x'^2 + y'^2 + z'^2 \quad = \quad x^2 + y^2 + z^2 = r^2$

LT: $(ct')^2 - x'^2 - y'^2 - z'^2 \quad = \quad (ct)^2 - x^2 - y^2 - z^2$

Space-Time Four-Vector

Definition — Space-time four-vector:

$$a \equiv (ct, x, y, z) = (ct, \boldsymbol{x})$$

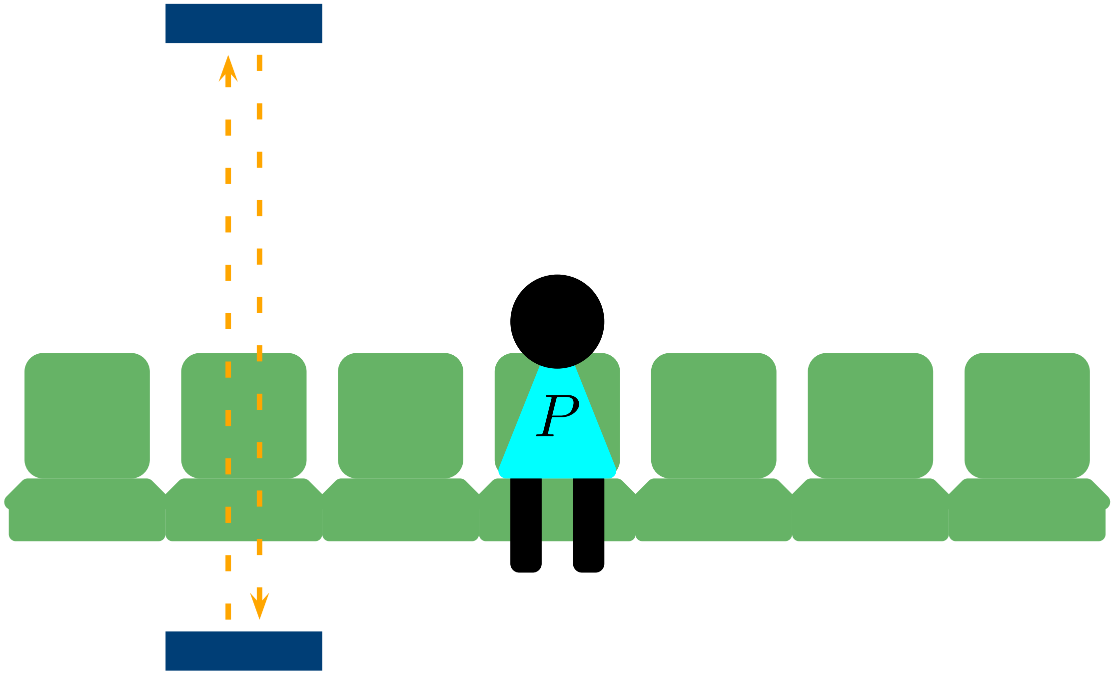
where $\boldsymbol{x}$ is the spacial three-vector.

Modulus:

$$a^2 \equiv (ct)^2 - x^2 - y^2 - z^2$$

a^2 is *invariant* under Lorentz transformations.

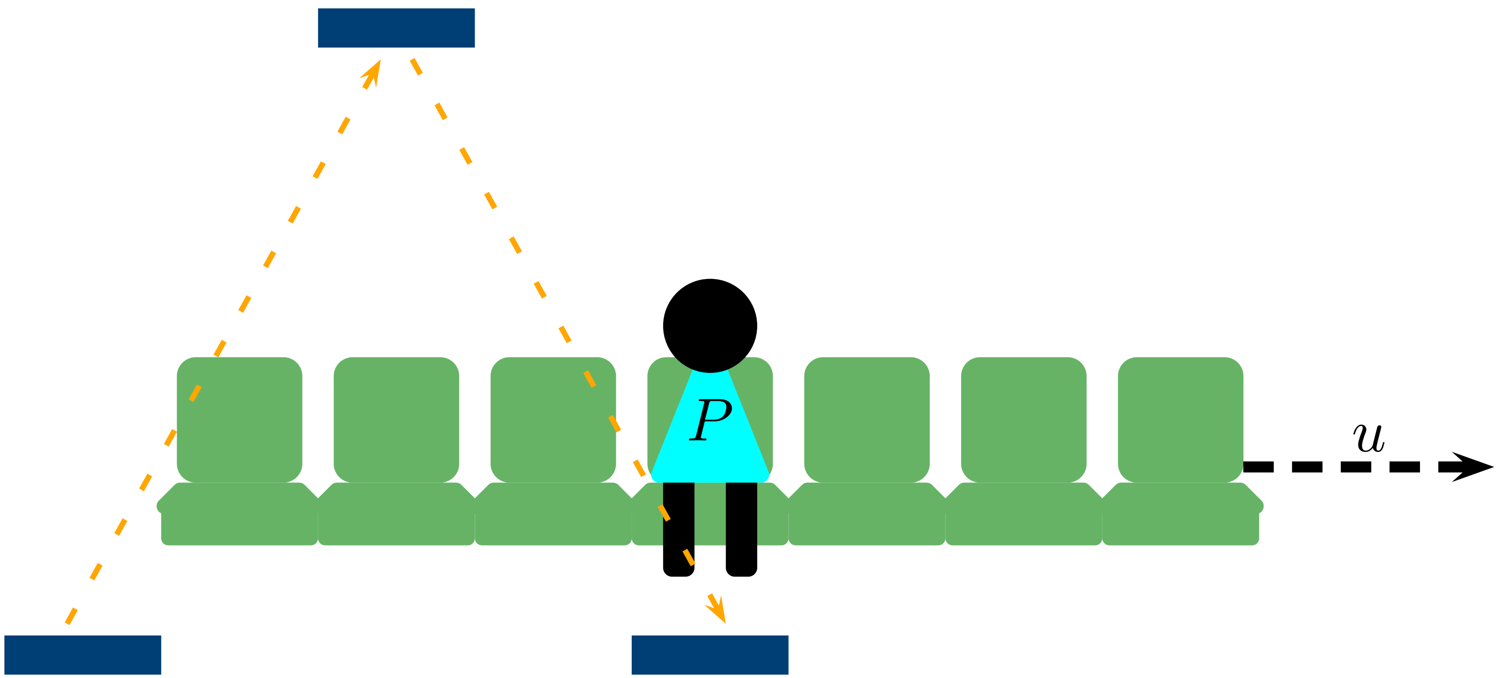
Clock on a Train



Assume $h = 2$ metres high. In $\mathcal{O}'$ the round-trip takes

$$s' = (4, 0, 0, 0) \quad [\mathrm{m}]$$
$$s'^2 = 4^2 \quad [\mathrm{m}^2]$$

Clock on a Train



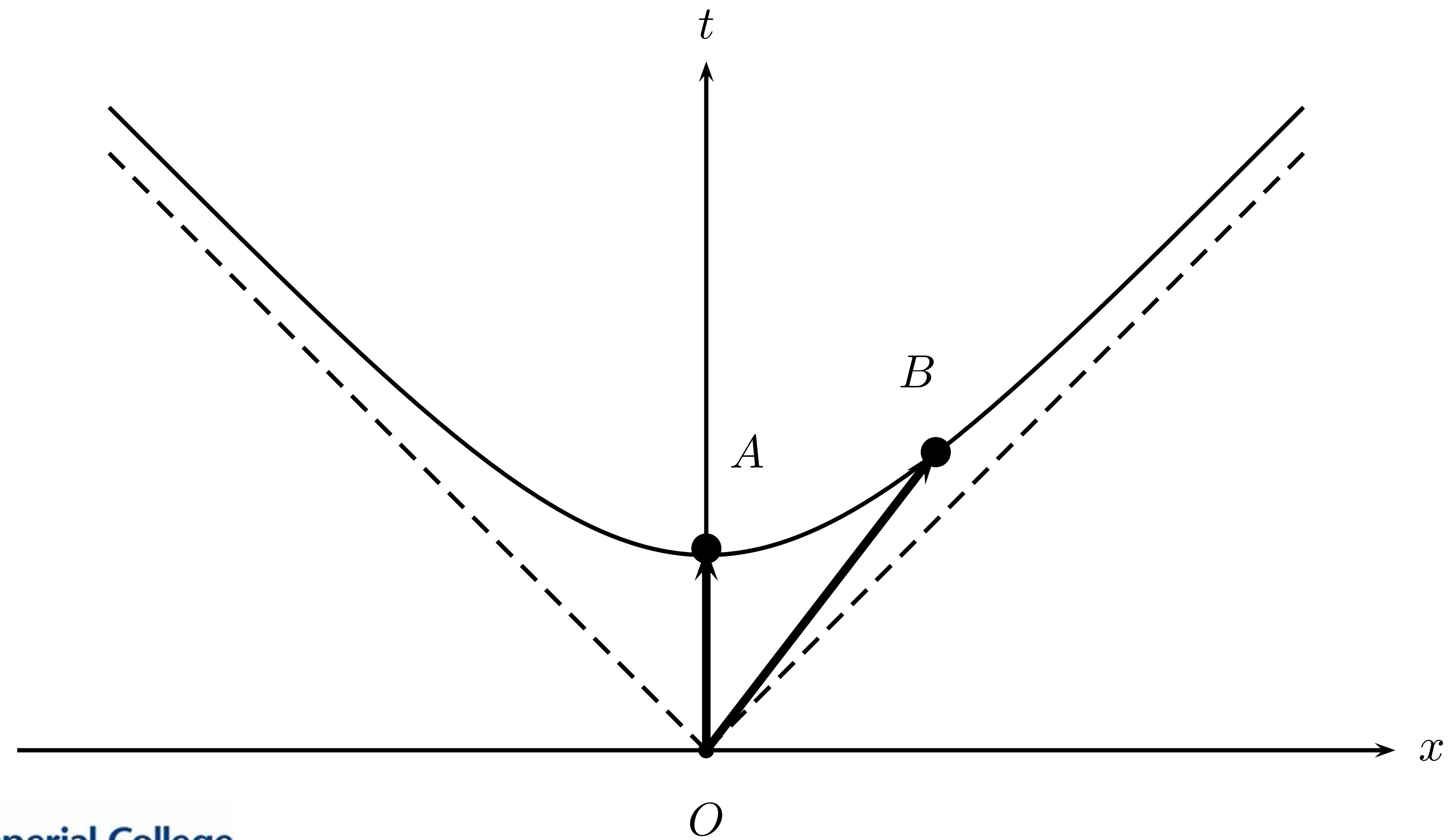
Assume $h = 2$ metres high. In $\mathcal{O}'$ the round-trip takes

$$\begin{aligned} s' &= (4,0,0,0) \quad [\mathrm{m}] \\ s'^2 &= 4^2 \quad [\mathrm{m}^2] \end{aligned}$$

In $\mathcal{O}$ frame the base moves by 3 metres. The trip is:

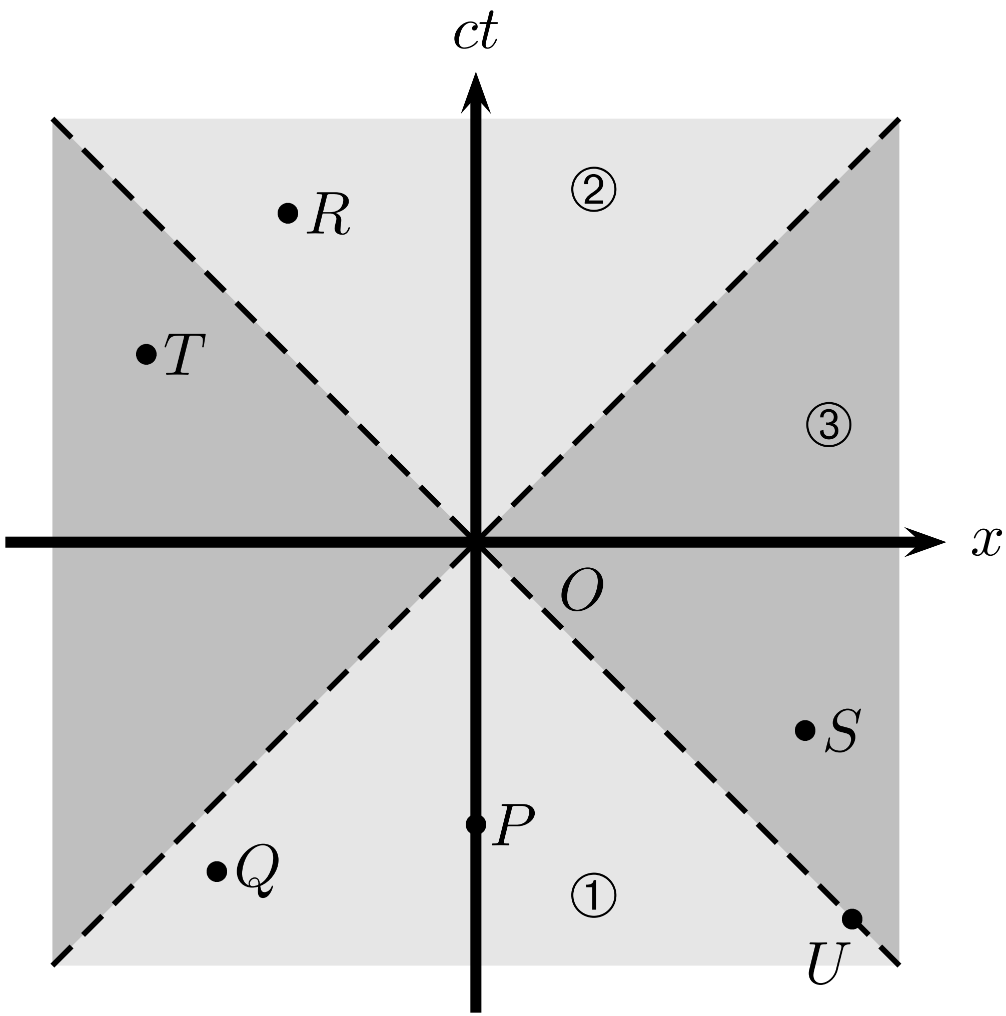
$$\begin{aligned} s &= (5,3,0,0) \quad [\mathrm{m}] \\ s^2 &= 5^2 - 3^2 = 4^2 \quad [\mathrm{m}^2] \end{aligned}$$

Invariance



Lecture 10

Spacetime Geometry

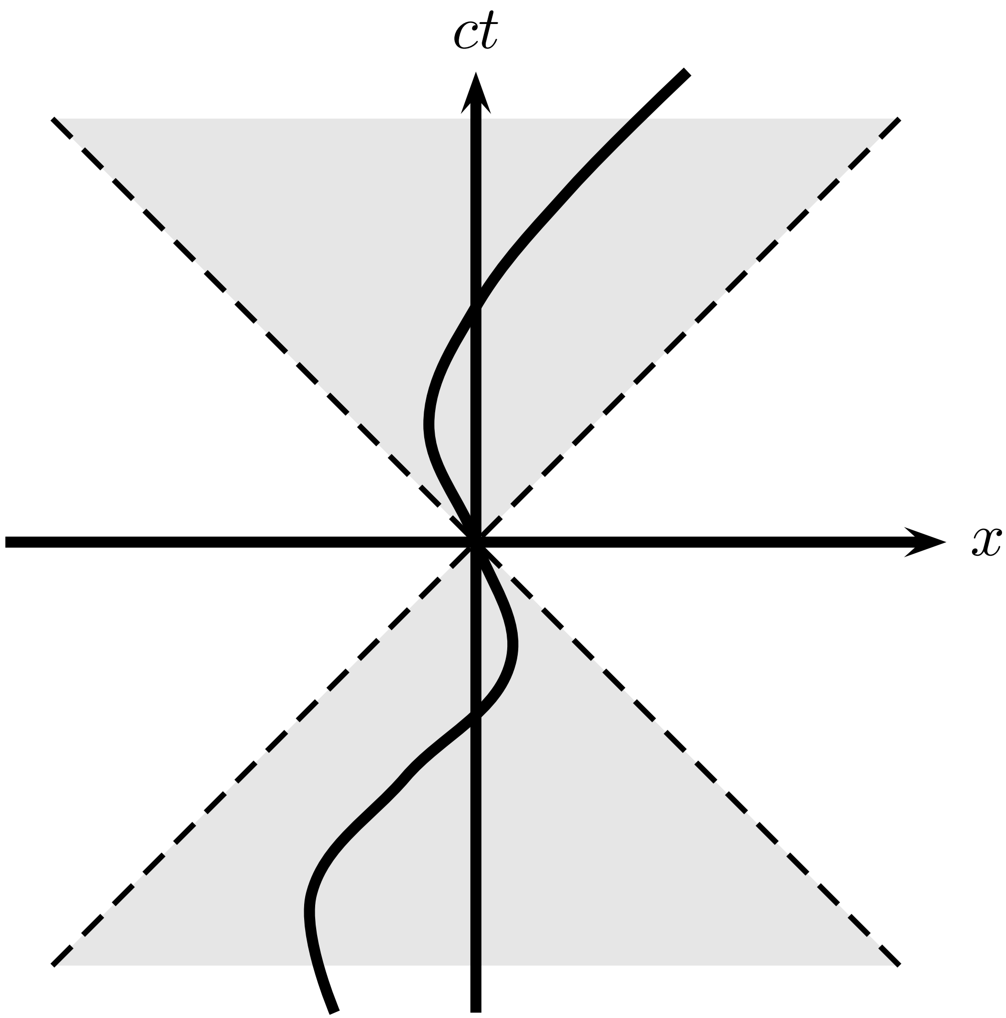


If $a^2 > 0$**,** a is called *timelike*,

If $a^2 < 0$**,** a is called *spacelike*,

If $a^2 = 0$**,** a is called *lightlike*.

Spacetime Geometry



If $a^2 > 0$**,** a is called *timelike*,

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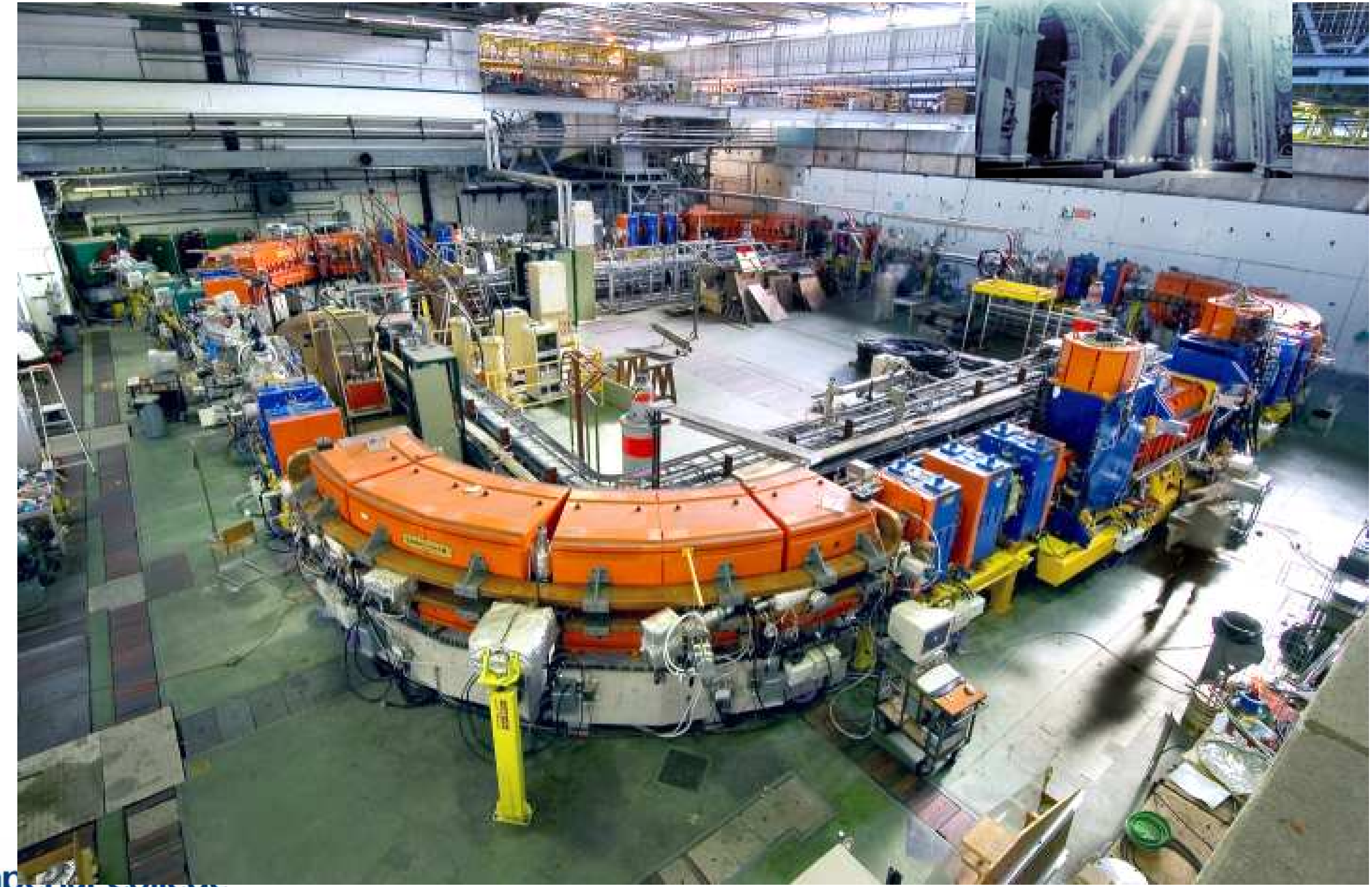
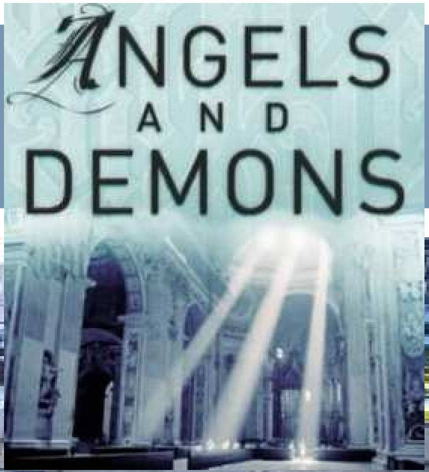
Time travel is not possible

CERN Antiproton Factory





Lecture Summary

What is Relativity?

Definition — Relativity:
Relativity is a theory describing the relation between observations (measurements) of the *same* process by *different* observers in motion *relative* to each other.

Special Relativity refers to the special case of *inertial* observers.

General Relativity refers to the general case of *accelerated* observers and provides a theory of gravity.

Postulates of Special Relativity

1. The laws of physics are identical in all inertial frames.
2. Light is propagated in empty space with a definite velocity c that is independent of the state of motion of the emitting body.

➜ The speed of light in vacuum has the same value c for all inertial observers.

$$c = 299,792,458 \text{ (exact) } \simeq 3 \cdot 10^8 \text{ m/s.}$$

Lorentz Transformations

$$\begin{aligned} x' &= \gamma(x - ut) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{ux}{c^2}\right) \end{aligned}$$

assuming $\mathcal{O}'$ moves at speed u along x relative to $\mathcal{O}$.

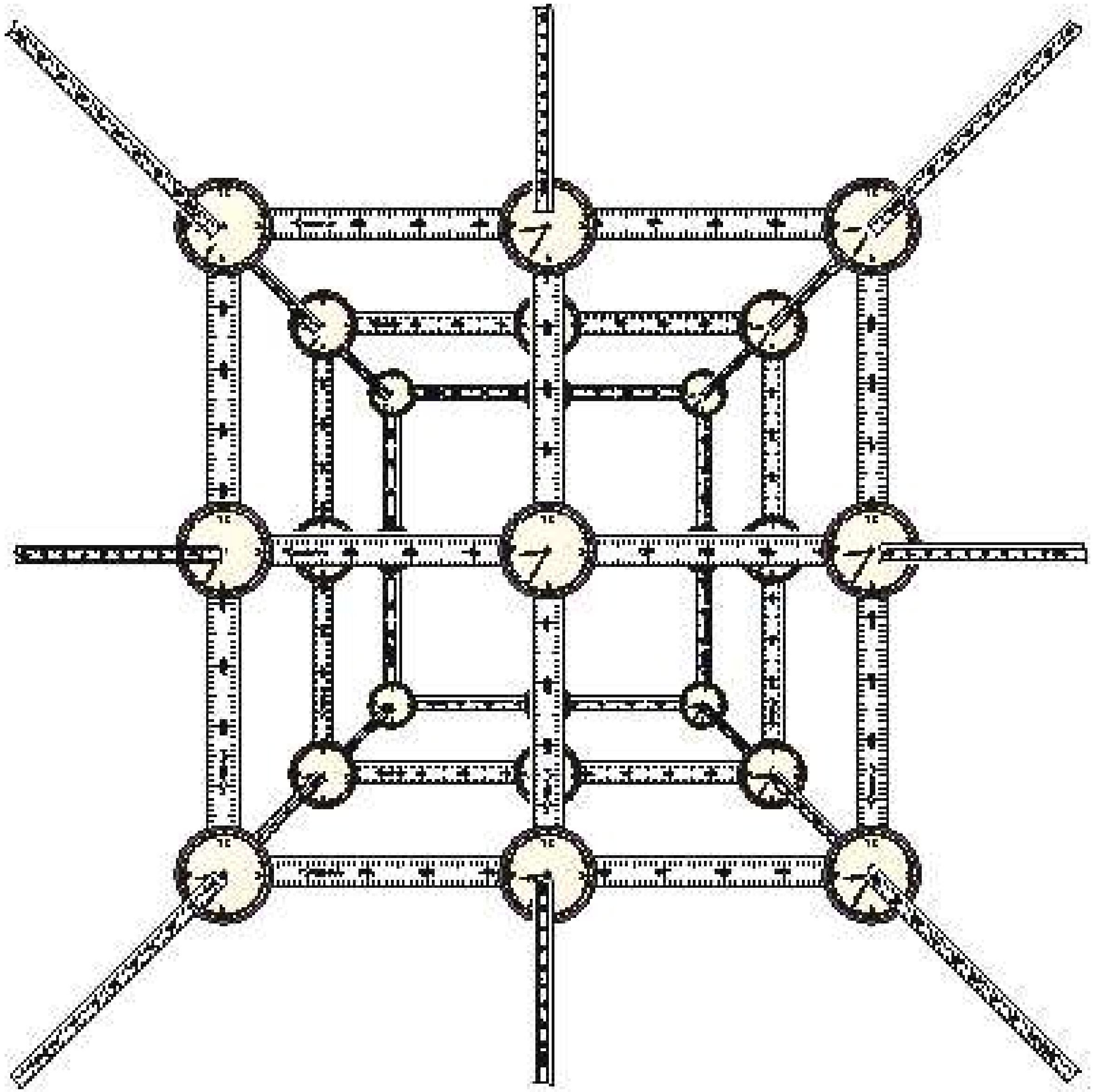
Inverse LT:

$$\begin{aligned} x &= \gamma(x' + ut') \\ y &= y' \\ z &= z \\ t &= \gamma(t' + \frac{ux'}{c^2}) \end{aligned}$$

Quote

Make things as simple as possible, but no simpler.

A Good Observer



Consequences of Relativity

Length contraction:

The measured length of a body is *greater* in its rest frame than any other frame.

Time dilation:

The measured time difference between the events represented by two readings of a given clock is *less* in the rest frame of the clock than in any other frame.

A body appears to be contracted, and time appears dilated, when seen from *another* frame.

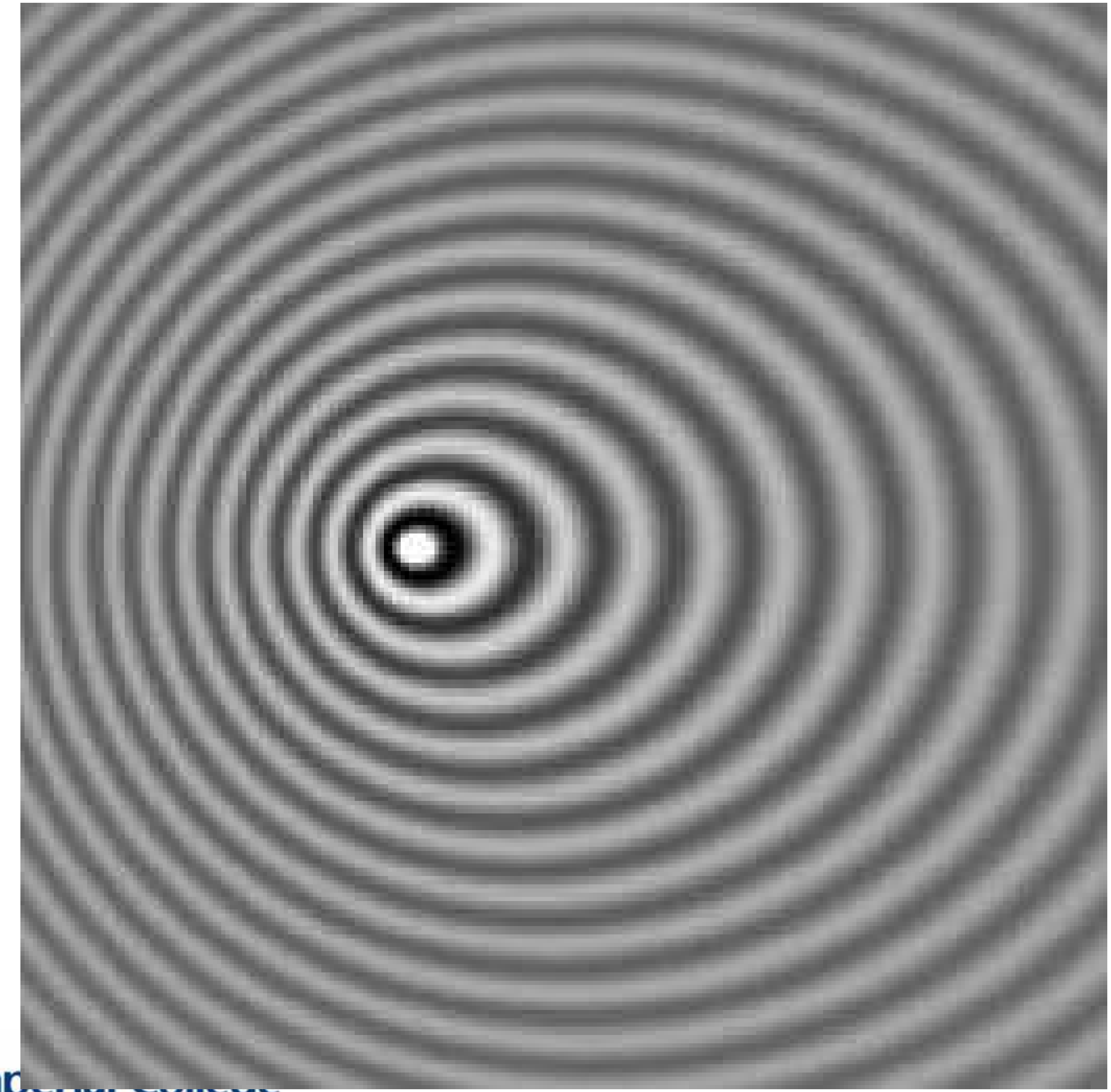
Relative motion

$$v'_x = \frac{v_x - u}{1 - \frac{u\, v_x}{c^2}}, \quad v'_y = \frac{v_y}{\gamma\left(1 - \frac{u\, v_x}{c^2}\right)}, \quad v'_z = \frac{v_z}{\gamma\left(1 - \frac{u\, v_x}{c^2}\right)}$$

The non-relativistic ($u \ll c$) limit is:

$$v'_x = v_x - u, \qquad v'_y = v_y, \qquad v'_z = v_z.$$

Relativistic Doppler Effect



$$\frac{f}{f_0} = \sqrt{\frac{1-\beta}{1+\beta}}$$

Energy and momentum

Definition — Momentum:

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Definition — Energy:

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Particle with Momentum p**:** $E^2 = p^2c^2 + m^2c^4$

Particle with Momentum $p = 0$**:** $E_0 = mc^2$

$$\beta = \frac{pc}{E}, \qquad \gamma = \frac{E}{mc^2}.$$

Lorentz Transforms

Lorentz Transformations ($\boldsymbol{x}, ct$)

$$
\begin{aligned}
x' &= \gamma\,(x - \beta ct) \\
y' &= y \\
z' &= z \\
ct' &= \gamma\,(ct - \beta x)
\end{aligned}
$$

Lorentz Transformations ($\boldsymbol{p}, E$)

$$
\begin{aligned}
p'_x &= \gamma\left(p_x - \beta\frac{E}{c}\right) \\
p'_y &= p_y \\
p'_z &= p_z \\
\frac{E'}{c} &= \gamma\left(\frac{E}{c} - \beta p_x\right)
\end{aligned}
$$

$\boldsymbol{p}$ transforms like x and E/c like ct.

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Energy-Momentum Four-Vector

Definition — Energy-momentum four-vector:

$$P \equiv \left(\frac{E}{c}, p_x, p_y, p_z\right) = \left(\frac{E}{c}, \boldsymbol{p}\right).$$

The scalar product of two four-momentum vectors is invariant:

$$P_1 \cdot P_2 \equiv \frac{E_1 E_2}{c^2} - \boldsymbol{p_1} \cdot \boldsymbol{p_2}.$$

With $P = P_1 = P_2$ we get

$$P^2 = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2 c^2,$$

Classwork

Twins paradox

- We have two twins, Al and Bob
- Bob embarks on a space-trip to Alpha Centauri at large speed. Al stays on Earth
- 10 years later Bob comes back and because of time contraction he is now younger than Al.

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- Who's right?

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- Who's right?

- The right question to ask: How many *inertial* reference frames are there in the problem?