December 13, 2007

Relativity Classwork 3

The Twin Paradox

3.1 Who's Younger?

The Twin Paradox is the most famous paradox of relativity. It has triggered a lot of literature, involving Einstein himself.

Suppose there are twins, Al and Bob, who of course have the same age. Bob embarks on a space exploration programme and is sent to Alpha Centauri which we'll assume to be 4 light-years away. Thanks to the progress of rocket science he travels at speed $\beta = \frac{4}{5}$. Without stopping on Alpha Centauri he returns to Earth at the same speed and meets his brother.

Because during the travel Bob's clock runs slow, considerably more time has passed in the earth frame ($\mathcal{A}$) than in the ship's frame ($\mathcal{B}$) and thus Al must be older than Bob.

But relativity says that all reference frames are equivalent, so from Bob's point of view he was at rest and Al moved at speed $\beta = -\frac{4}{5}$ and thus he must be younger.

How can one reconcile time contraction and equivalence of reference frames? Is there really a younger twin, and if yes, who?

3.2 Some numbers

Assume Bob started at time $t = t' = 0$ (in frames $\mathcal{A}$ and $\mathcal{B}_1$, respectively) from position $x = x' = 0$ on a 25^{th} of December. Write the following events in both frames. Use light-years as unit for space and years for time, such that $c = 1$ in these units.

1. Bob's start.
2. Bob's arrival on Alpha Centauri.
3. Bob's return on Earth.

How much time elapsed for both twins between 1 and 3? Check if $c^2t'^2 - x'^2 = c^2t^2 - x^2$. What does it mean if it's not conserved?

3.3 Christmas Cards

Al and Bob send each other e-cards every year on Christmas day. These are transmitted at speed of light and so they used to get them almost immediately while on Earth. They continue with this tradition during Bob's trip, but of course the cards will not be received on Christmas day anymore. In the following we'll only consider the cards Al sends to Bob, but you can repeat the exercise for Bob's cards.

Al gives a card to Bob just before he leaves and then sends one card every Christmas according to his calendar.

1. How many does Al send?
2. When and where does Bob get them, as seen in Al's frame?
3. When does Bob get them, as seen in his frame? Be careful that Lorentz transforms can only be done for frames that coincide at $t = t' = 0$.
4. How many cards does Bob get?
5. At what frequency does Bob get the cards on his two trips? Does it match the expectation from the Doppler effect?
6. Give an example of two events which do not happen in the same order in Al and Bob's frames.