4 Relativistic Mechanics

We have discussed velocities will now define relativistic momentum and energy. The most important applications of relativistic mechanics is in high-energy particle interactions, for instance collisions or decays. We shall mainly focus on those.

But first let's start with our light clock.

4.1 Energy of the Light Clock

Suppose we have a light clock at rest and free to float. What would happen when we switch it on and the first pulse is emitted?

We know the energy of photons from Planck's law

$$E = h\nu$$

and the momentum from de Broglie[5]

$$p = \frac{h}{\lambda} \quad \text{with} \quad c = \nu\lambda \quad \Rightarrow \quad p = \frac{h\nu}{c} = \frac{E}{c}. \tag{21}$$

A light clock of mass M and length L emits a first pulse of energy E and momentum E/c. By momentum conservation the clock recoils at a speed

$$v = -\frac{p}{M} = -\frac{E}{Mc},$$

where we assume $v \ll c$. After travelling for a time $\Delta t \simeq L/c$ the radiation hits the other end of the clock which brings it back to rest again (Fig. 21). In the process the box has moved by

$$\Delta x = v\Delta t = -\frac{EL}{Mc^2},$$

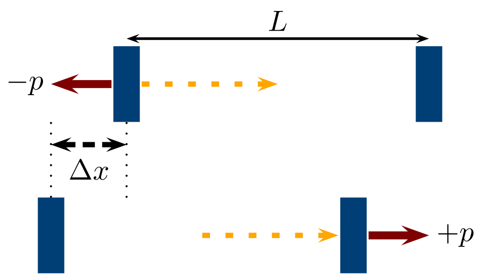

Figure 21: Momentum conservation in a light clock.

where we used $v \ll c$ again. Does this mean the centre of mass of the clock has moved? This cannot be. As no external force acts on the clock the centre of mass cannot have moved. To keep the centre of mass in place we need to *postulate* that the burst of light has transferred some mass m from the left to the right part of the clock such that

$$mL + M\Delta x = 0.$$

This postulated mass can then be calculated as

$$m = -\frac{M}{L}\Delta x = \frac{M}{L}\frac{EL}{Mc^2} = \frac{E}{c^2},$$

or

$$E = mc^2, \tag{22}$$

which you might have seen before.

What does that mean? It is *not* the mass of the light burst. Light has no mass. It is the amount of mass which has been lost by the left side of the clock and absorbed by the right side in the process.

[5]We know this but Einstein didn't in 1905 as de Broglie's law dates from 1924 (and Planck's from 1900). He reached the same conclusion in a very different way in [15].

Einstein postulated in his second relativity article [15] that any change of energy in the reference frame of a body is associated to a change of mass. The left side of the clock becomes lighter by a mass E/c^2 and the right side heavier. Mass is not conserved in a process involving a change of energy.

It looks as if we haven't used any of the postulates of special relativity here and that Newton's laws could have predicted the same result. This is not really true as we have implicitly used that light is emitted as speed c and that it propagates at speed c.[6]

This result postulates the equivalence of mass and energy. Mass is just a form of energy at rest. Note that it is a very dense form of energy: c^2 is a huge factor. The energy produced by a 1 GW power plant in a year is equivalent to $350\,\mathrm{g}$ of mass.

Let's now define more precisely the relativistic momentum and energy.

4.2 Energy and Momentum Conservation

L.7

In special relativity energy and momentum conservation is actually more useful than the relativistic form of Newton's second law.

The classical momentum conservation in an elastic collision says:

$$m_1\boldsymbol{v}_{\mathbf{1}}^{\text{in}} + m_2\boldsymbol{v}_{\mathbf{2}}^{\text{in}} = m_1\boldsymbol{v}_{\mathbf{1}}^{\text{out}} + m_2\boldsymbol{v}_{\mathbf{2}}^{\text{out}}.$$

But this is not covariant under LT. (Problem 2.3).

We need to redefine the momentum to preserve the law. We need

1. A definition of the momentum p such that it is conserved.
2. The low-speed limit must be $\boldsymbol{p} = m\boldsymbol{v}$.
3. The conservation laws must be covariant under LT.

We the postulate

Definition — Momentum:

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m\boldsymbol{v} \tag{23}$$

Does this definition address the constraints listed above?

1. The conservation has to be tested experimentally.
2. The low-speed limit is $p = mv$ if $v \ll c$.
3. If $\boldsymbol{p_1} + \boldsymbol{p_2} =$ constant, is $\boldsymbol{p_1'} + \boldsymbol{p_2'} =$ constant$'$ when $\boldsymbol{p_i}$ and $\boldsymbol{p_i'}$ are related by a LT?

Let's simplify by having a momentum along x, i.e. $\boldsymbol{p} = (p, 0, 0)$. Using Eq. (23) we have

$$\begin{aligned} p' &= \frac{mv'}{\sqrt{1-\frac{v'^2}{c^2}}} \quad \text{with} \quad v' \overset{(14)}{=} \frac{v-u}{1-\frac{uv}{c^2}} \\ &= \frac{mv-mu}{1-\frac{uv}{c^2}} \frac{1}{\sqrt{1-\frac{(v-u)^2}{c^2\left(1-\frac{uv}{c^2}\right)^2}}} = \gamma p - \gamma\frac{u}{c^2}\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}, \end{aligned}$$

[6] Although Newton himself wrote *Are not gross bodies and light convertible into one another [...]?* in 1730.

after some algebra. Thus with $p = p_1 + p_2$ we get

$$p_1' + p_2' = \underbrace{\gamma\,(p_1 + p_2)}_{\text{constant}} - \underbrace{\gamma\frac{u}{c^2}}_{\text{constant}} \left(\frac{m_1 c^2}{\sqrt{1-\frac{v_1^2}{c^2}}} + \frac{m_2 c^2}{\sqrt{1-\frac{v_2^2}{c^2}}} \right).$$

The momentum is conserved provided the quantity

$$\frac{m_1 c^2}{\sqrt{1-\frac{v_1^2}{c^2}}} + \frac{m_2 c^2}{\sqrt{1-\frac{v_2^2}{c^2}}}$$

is also conserved.

This is not different from Galilean relativity (Eq. (10)) where momentum is conserved provided that mass is conserved. Here we have no requirement on mass (we know it's related to energy) but on the quantity above. We shall define as the energy:

Definition — Energy:

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m c^2 \tag{24}$$

So momentum conservation applies in all inertial frames, *if* energy is conserved. Here again there is no mathematical proof that this quantity is indeed conserved. It's an experimental fact.

4.3 Energy-Momentum Relations

In Newtonian mechanics we relate the kinetic energy K to the momentum by

$$K = \frac{p^2}{2m}.$$

How is this relation in relativistic mechanics? Using Eq. (23) and (24) we get

$$\frac{p}{E} = \frac{mv}{mc^2} = \frac{v}{c^2} \quad \Rightarrow \quad v = \frac{p}{E}c^2, \tag{25}$$

and

$$E^2 = \frac{m^2c^4}{1-\frac{v^2}{c^2}} = \frac{m^2c^4}{1-\frac{p^2}{E^2}c^2} = \frac{E^2m^2c^4}{E^2-p^2c^2} \quad \Rightarrow \quad E^2 - p^2c^2 = m^2c^4$$

or

$$E^2 = p^2c^2 + m^2c^4, \tag{26}$$

which is the relativistic energy-momentum relation.

4.3.1 Rest Energy

An object at rest has $p = 0$ and therefore

$$E_0 = mc^2. \tag{27}$$

A particle at rest has an energy equal to its mass (times some constant). The mass at rest ("rest mass" or "proper mass") is a property of the particle.

4.3.2 Kinetic Energy

If $E_0 = mc^2$ is the rest energy one can define the kinetic energy as

$$K = E - E_0 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2 = mc^2\,(\gamma - 1) \tag{28}$$

4.3.3 Non-relativistic approximation

At low speeds $v \ll c$ we get the momentum

$$p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \Rightarrow \qquad p \simeq mv$$

and the kinetic energy

$$K = mc^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right) = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots - 1\right) \simeq \frac{1}{2}mv^2,$$

and recover Newton's quantities.

4.3.4 Ultra-relativistic approximation

At very high speeds $v \simeq c$ (but smaller!) the rest energy of the particle is negligible compared to the total energy. We then have

$$E^2 = p^2c^2 + m^2c^4 \simeq p^2c^2 \qquad \Rightarrow \qquad E \simeq pc.$$

This is a very good approximation for the muon in Section 3.6.1.

4.3.5 About Photons

Where does the photon fit into all this? We know from Planck and de Broglie that $E = pc$ (Eq. (21)). But we also know $E^2 = p^2c^2 + m^2c^4$, so the photon must have no mass. Using Eq. (25):

$$v = \frac{p}{E}c^2 = c$$

Photons travel at speed of light at any energy. What differs is their frequency, i.e. their colour.

4.3.6 High-Energy Electrons

In an X-ray gun electrons are accelerated by an electric potential of $U \sim 10^6$ V. What's the electron's speed?

Newton would have said

$$K = eU = \frac{1}{2}mv^2$$

with $m_\mathrm{e} \sim 10^{-30}$ kg and $e = 1.6 \cdot 10^{-14}$ C,

$$v^2 = \frac{2eU}{m_\mathrm{e}} \simeq \frac{2 \cdot 1.6 \cdot 10^{-14} \cdot 10^6}{10^{-30}} = 32 \cdot 10^{16} \, \frac{\mathrm{m}^2}{\mathrm{s}^2} \qquad \Rightarrow \qquad v \simeq 6 \cdot 10^8 \, \frac{\mathrm{m}}{\mathrm{s}} \simeq 2c,$$

clearly an invalid use of Newtonian mechanics. We have comparable kinetic energy $K = 16 \cdot 10^{-14}$ J and rest mass energy $m_\mathrm{e}c^2 = 9 \cdot 10^{-14}$ J.

The correct way is to use Eq. (25)

$$v = \frac{p}{E}c^2 \quad \text{with} \quad E = K + m_\mathrm{e}c^2 = 25 \cdot 10^{14} \text{ J}$$

and the momentum is obtained from Eq. (26)

$$p^2c^2 = E^2 - m_\mathrm{e}^2c^4 \qquad \Rightarrow \qquad \frac{pc}{E} = \sqrt{1 - \left(\frac{m_\mathrm{e}c^2}{E}\right)^2} = \frac{v}{c}$$

and therefore

$$v = c\sqrt{1 - \left(\frac{9}{25}\right)^2} \simeq 0.9c.$$

It takes a million volts to accelerate an electron to 90% of the speed of light.

4.4 Some Useful Relations

We now have some useful relations between E, p, β and γ:

For a particle of mass m, momentum p and total energy E

$$\beta = \frac{pc}{E}, \qquad \gamma = \frac{E}{mc^2}. \tag{29}$$

5 Applications

5.1 Units

Charged particles attain high energies via electric fields, as in the example above. For fundamental particles of charge $\pm e$, the best is to measure the energy in units of fundamental charge times electric potential difference.

The kinetic energy of a particle of charge e accelerated by 1V is $1 \text{ eV} = 1.6 \cdot 10^{-19}$J. The use of multiples of eV is more practical for atomic or sub-atomic physics than Joules (see Problem 2.4).

Energies are measured in eV (typical energies of atoms in gas at room temperature), keV (kinetic energy of electrons in an old-style TV set), MeV (as the electron above), GeV (at big accelerators), and even TeV ($= 10^{12}$ eV $\simeq 10^{-7}$ J, at the biggest accelerators like the Large Hadron Collider at CERN).

Masses are then measured in units of eV/c^2. For instance

$$1\ \mathrm{GeV}/c^2 = 10^9 \frac{1\mathrm{V}\cdot e}{c^2} = 10^9\ \frac{1.6\cdot 10^{-19}}{9\cdot 10^{16}} \simeq 2\cdot 10^{-27}\ \mathrm{kg}$$

is about the mass of the proton. The electron "weights" $511\ \mathrm{keV}/c^2$.

Similarly one measures momenta in eV/c.

Since the factors c and e are absorbed into the units, one can write Eq. (26) as

$$E^2 = p^2 + m^2$$

when using these units. This avoids clutter of c's and e's in equations, like in our electron speed calculation above.

5.1.1 High-Energy Electrons Revisited

Using these relations we solve the problem in Section 4.3.6 in one line

$$\beta = \frac{p}{E} = \frac{\sqrt{E^2 - m^2}}{E} = \frac{\sqrt{1-\left(\frac{1}{2}\right)^2}}{1} \simeq 0.9.$$

5.2 Nuclear Physics

L.8

A nucleus is a *bound* state of *nucleons* (p,n). Where bound means that

$$E_{\text{Nucleus}} < E_{\text{free constituents}}$$

else it would be advantageous to be free and the nucleus would fall apart. Since rest energy is related to mass, we have

$$m_{\text{Nucleus}} < \sum m_{\text{constituents}}.$$

Example: The simplest nucleus is the deuterium $\mathrm{D} = {}^2_1\mathrm{H} = (\mathrm{p},\mathrm{n})$. The masses are [14]:

$$\begin{aligned} m_\mathrm{p} &= 938.272\ \mathrm{MeV}/c^2 \\ m_\mathrm{n} &= 939.565\ \mathrm{MeV}/c^2 \\ m_\mathrm{D} &= 1875.613\ \mathrm{MeV}/c^2 \end{aligned}$$

The difference in energies $(m_\mathrm{D} - m_\mathrm{p} - m_\mathrm{n})c^2 = 2.22\ \mathrm{MeV}$ is the *binding energy* of the deuteron nucleus. This means the fusion of a proton and a neutron would free about 2 MeV of energy in form of high-energy photons. However, free neutrons are not available in nature as they decay.

But fusion of deuterium works. In the sun the fusion

$$\mathrm{D} + \mathrm{D} \Rightarrow {}^3_2\mathrm{He} + \mathrm{n} + 3\ \mathrm{MeV}$$

happens all the time and is providing the fuel for life on earth.

Fusion is possible when the binding energy per nucleon increases with the number of nucleons in the nucleus. Fission is possible when this energy decreases. The binding energy is shown for all nuclei in Figure 22. At the top of the curve is iron $^{56}\mathrm{Fe}$, which is the most bound nucleus. Nuclear fusion in stars stops at iron, that's why there is so much iron around as compared to similar atoms. The earth core is mainly made of iron.

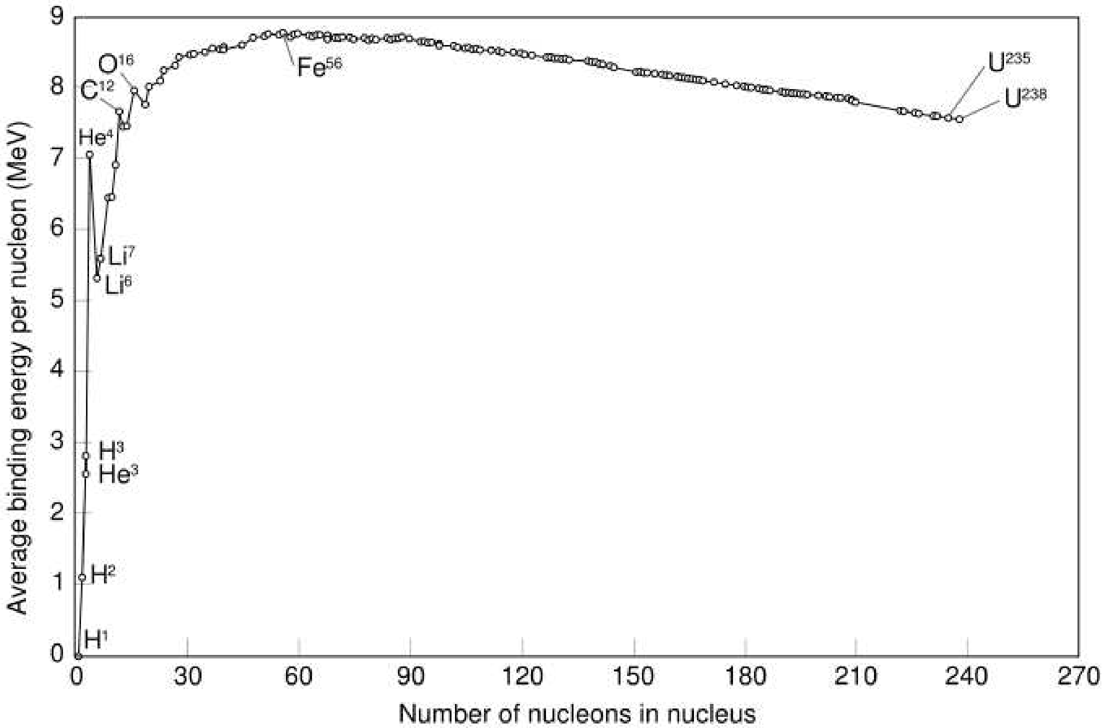

Figure 22: Binding energy of various nuclei.

5.3 Particle Collisions

Momentum and energy conservation in collisions and decays is written

$$\left.\begin{array}{rcl} \sum\limits_i \boldsymbol{p}_i & = & \sum\limits_o \boldsymbol{p}_o \\ \sum\limits_i \boldsymbol{E}_i & = & \sum\limits_o \boldsymbol{E}_o \\ \text{with } E^2_{(i,o)} & = & p^2_{(i,o)}c^2 + m_{(i,o)}c^4 \end{array}\right\} \quad (30)$$

where i stands for *incoming* and o for *outgoing*. In an elastic collision the same particles appear on both sides of the equations, in a decay there is only one incoming particle, the general case being the inelastic collision where there can be any number of particles on each side. The values of $\boldsymbol{p}$ and E differ for observers in different inertial frames, but the relations above remain valid in *any* inertial frame. This is the *covariance of energy and momentum relations*.

A few examples are given below.

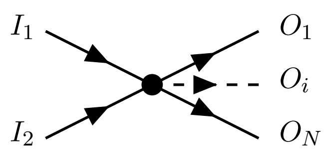

Figure 23: Collision, decay...

5.3.1 Particle Decay at Rest

Take the decay of the neutral kaon to two pions $\mathrm{K}^0 \to \pi^+\pi^-$ in its rest frame.

$$\begin{array}{rl} \text{Momentum conservation:} & 0 = \boldsymbol{p}_1 + \boldsymbol{p}_2 \\ & p_1 = p_2 \quad (= p_\pi, \text{ say}) \\ \text{Energy conservation:} & E_{\mathrm{K}^0} = m_{\mathrm{K}^0}c^2 = E_1 + E_2 \end{array}$$

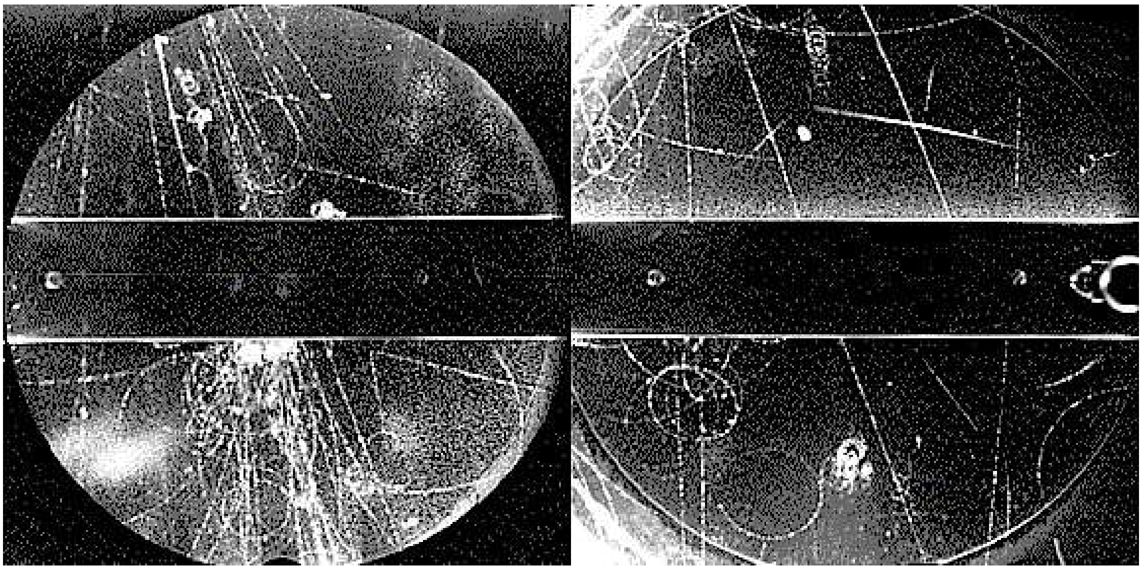

Figure 24: Decays of a K^0 (left) and a K^+ (right), Manchester, December 20, 1947 [16].

where

$$E^2_{(1,2)} = p^2_\pi c^2 + m^2_\pi c^4 \quad \Rightarrow \quad E_1 = E_2 \quad (= E_\pi, \text{say}).$$

Experiment gives

$$\begin{aligned} m_{K^0} &= 498\,\text{MeV}/c^2 \quad \Rightarrow \quad E_\pi = \frac{1}{2} m_{K^0} c^2 = 249\,\text{MeV} \\ m_\pi &= 139\,\text{MeV}/c^2 \end{aligned}$$

and

$$p^2_\pi c^2 = E^2_\pi - m^2_\pi c^4 = 249^2 - 139^2 \quad \Rightarrow \quad p_\pi = 206\,\text{MeV}/c.$$

The two pions fly away with momenta of $206\,\text{MeV}/c$ and speeds of $\beta = \frac{206}{248} \simeq 0.7$.

This process is purely relativistic. Newton's laws do not allow mass to be converted into kinetic energy.

5.3.2 Particle Decay in Flight

In the real world a K^0 is almost never at rest. It is flying with a momentum $\boldsymbol{p}_{K^0}$ in the laboratory frame $\mathcal{O}$, as in Figure 24. But we know the analysis is simple in the K^0 frame $\mathcal{O}'$: $\boldsymbol{p}'_1 = \boldsymbol{p}'_2 = 206\,\text{MeV}/c2$. So we simply need to Lorentz-transform this result into the laboratory frame.

To do this we need Lorentz transformations for momenta.

5.3.3 Lorentz Transformations for Momentum and Energy

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We want to transform p to p' using the velocity transformations v to v' given in Eq. (14):

$$p'_x = \gamma p_x - \frac{\gamma u}{c^2} \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \left(p_x - \frac{uE}{c^2} \right)$$

a similar analysis leads to the transformations:

$$\left.\begin{array}{rcl} p'_x & = & \gamma\left(p_x - \frac{uE}{c^2}\right) \\ p'_y & = & p_y \\ p'_z & = & p_z \\ E' & = & \gamma\left(E - up_x\right) \end{array}\right\} \qquad (31)$$

where γ involves the velocity u of the frame $\mathcal{O}'$ relative to $\mathcal{O}$.

We then get the Lorentz transformations:

Lorentz Transformations ($\boldsymbol{x}, ct$) **Lorentz Transformations ($\boldsymbol{p}, E$)**

$$\left.\begin{array}{rcl} x' & = & \gamma\left(x - \beta ct\right) \\ y' & = & y \\ z' & = & z \\ ct' & = & \gamma\left(ct - \beta x\right) \end{array}\right\} \quad (32) \qquad \left.\begin{array}{rcl} p'_x & = & \gamma\left(p_x - \beta\frac{E}{c}\right) \\ p'_y & = & p_y \\ p'_z & = & p_z \\ \frac{E'}{c} & = & \gamma\left(\frac{E}{c} - \beta p_x\right) \end{array}\right\} \quad (33)$$

$\boldsymbol{p}$ transforms like $\boldsymbol{x}$ and E/c like ct.

Inverse transformations also take the usual form.

5.3.4 Particle Decay in Flight

Back to our K^0. Consider a decay along the x-axis, the axis of motion of the K^0. From Section 5.3.1 we have

$$\begin{array}{rclcrcl} (p'_1)_x & = & 206\ \mathrm{MeV}/c & \qquad & (p'_2)_x & = & -206\ \mathrm{MeV}/c \\ E'_1 & = & 249\ \mathrm{MeV}/c & & E'_2 & = & 249\ \mathrm{MeV}/c \end{array}$$

Let's transform this to the laboratory frame $\mathcal{O}$ using inverse LT:

$$(p_1)_x = \gamma\left[(p'_1)_x + \beta\frac{E'_1}{c}\right]$$

with $\beta = u/c$ and u is the speed of $\mathcal{O}'$ relative to $\mathcal{O}$, i.e. the speed of the K^0 relative to the laboratory, which following Eq. (25) is

$$u = \frac{p_{K^0}}{E_{K^0}}c^2.$$

For example a K^0 with $p_{K^0} = 500\ \mathrm{MeV}/c$ in the laboratory has

$$\begin{array}{rcl} E_{K^0} & = & \sqrt{p_{K^0}^2c^2 + m_{K^0}c^4} \simeq 700\ \mathrm{MeV} \\ \beta & = & \dfrac{p_{K^0}c}{E_{K^0}} \simeq \dfrac{5}{7} \qquad \gamma = \dfrac{E_{K^0}}{m_{K^0}c^2} \simeq \dfrac{7}{5} \\ (p_1)_x & = & \dfrac{7}{5}\left[+206 + \dfrac{5}{7}249\right] \simeq 539\ \mathrm{MeV}/c \\ (p_2)_x & = & \dfrac{7}{5}\left[-206 + \dfrac{5}{7}249\right] \simeq -39\ \mathrm{MeV}/c, \end{array}$$

and we have one pion going forward at a large speed ($\beta = 0.98$) and the other one backward with a low speed ($\beta = 0.27$).