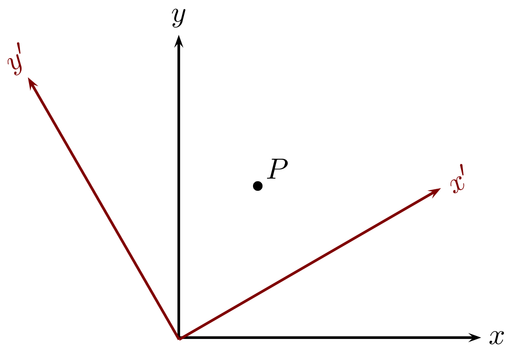


Figure 25: Rotation in 2D.

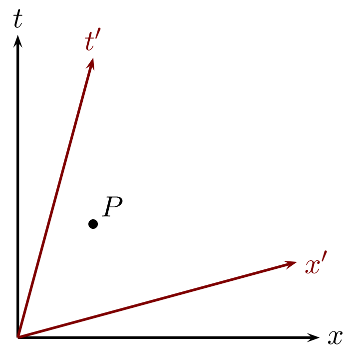


Figure 26: Lorentz-transform.

# 6 Four-Vectors

L.9

Lorentz transformations teach us that there is no absolute space and no absolute time. Time and space mix depending on the speed of the observer. Each observer knows what his space and his time is and doesn't feel like they mix, but when talking to another observer in another frame of reference, one has to apply Lorentz transforms, in particular

$$x' = \gamma\,(x - \beta ct) \qquad ct' = \gamma\,(ct - \beta x)\,. \tag{34}$$

$x'$ is an admixture of $x$ and $t$ and $t'$ an admixture of $t$ and $x$.

That's similar to rotations in two dimensions. If you speak about the position of a point $P$ with an observer whose reference frame is rotated by an angle $\alpha$ with respect to yours you will have to apply the transformation

$$x' = x\cos\alpha + y\sin\alpha \qquad y' = y\cos\alpha - x\sin\alpha, \tag{35}$$

as in Figure 25. $x'$ is an admixture of $x$ and $y$.

Is a Lorentz transform just a rotation is space-time? Not really, the key problem being the presence of the two minus signs in the Lorentz transform while there is a minus and a plus in the rotation. See Figure 26 for a graphical representation.

But there are similarities. In two dimensions there is one obvious quantity which is conserved by rotations: the distance $r$:

$$r'^2 = x'^2 + y'^2 = x^2 + y^2 = r^2,$$

which you can easily prove using Eq. (35). $r$ is an invariant of 2D-rotations and can easily be generalised to 3D-rotations

$$r'^2 = x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 = r^2. \tag{36}$$

Is there an equivalent for Lorentz Transforms? Yes, it is

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{37}$$

Note the minus signs! The proof is in Problem 3.1.

Given this invariant it is quite natural to extend three-dimensional vectors to four-vectors including time. They are usually written as

**Definition — Space-time four-vector:**

$$a \equiv (ct, x, y, z) = (ct, \boldsymbol{x})$$

where $x$ is the spacial three-vector.

Some authors write them with the space component first and the time last. They can also be written in columns. Finally some write the first term as $-ct$. It's just a matter of conventions and does not matter as long as one gets Eq. (38) and (40) below right. There is also no standard notation for four-vectors unlike three-vectors which are shown in bold-type. In these notes we shall always call a space-time four-vector $a$ or $b$.

Unlike for three-vectors we shall define the squared modulus as follows:

**Definition — Squared Modulus of a four-vector:**

$$a^2 \equiv ct^2 - x^2 - y^2 - z^2 \tag{38}$$

$a^2$ is *invariant* under Lorentz transformations as shown in Eq. (37).

So for any four-vector $a$ and any Lorentz transformation $LT$:

$$a^2 = [LT(a)]^2 \,. \tag{39}$$

Students who like linear algebra might try to write $LT$ as a $4 \times 4$ symmetric square matrix (Problem 3.4). But we won't need this.

## 6.1 Example

For example go back to Figures 11 and 12 on p. 11. Consider the two events

- Emission of a ray of light from the base of the clock, (which we set to be the origin $O$)
- Reception of this ray back on the base.

In the first case (Fig. 11) we have the light clock at rest in the train. If we suppose the clock is $h = 2$ metres high, the light will take a time $ct' = 4\,\mathrm{m}$ for the round trip. The four-vector is then

$$s' = (4, 0, 0, 0) \quad [\mathrm{m}].$$

In the frame of the earth the clock is moving with the train. Suppose the base has moved by $3\,\mathrm{m}$ during the time the light travels in the clock. At the end of the round trip the base will be at $x = 3\,\mathrm{m}$. And the time taken is given by Pythagoras: $(ct)^2 = x^2 + (2h)^2 = 5^2$.

The four-vector in the earth frame is

$$s = (5, 3, 0, 0) \quad [\mathrm{m}].$$

And we have

$$s^2 = 5^2 - 3^2 = 4^2 = s'^2 \quad [\mathrm{m}^2].$$

Figure 27 shows a 2D projection of four dimensional space-time where the two space-time vectors are shown. The round-trip trajectory of the ray of light in the frame of the train is $OA \equiv s'$, with $x = 0$ staying constant. In the frame of the earth the light ray goes forward in $x$ and takes longer in $t$: this is the $OB \equiv s$ vector. Both are on a hyperbola at constant $a^2$. The dashed diagonals show trajectories at speed of light (so their slope is $\mathrm{d}x/\mathrm{d}t = c$). They are the limit for train speeds close to the speed of light.

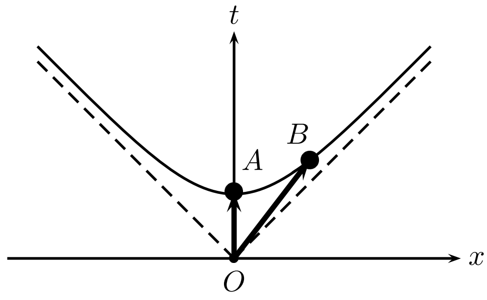


Figure 27: Graphical representation of the invariance.

The fact that the curve is a hyperbola is a consequence of the minus sign in the modulus of four-vectors. In the case of the 3D distance invariance $r^2$ we would draw a circle.

## 6.2 Space-Time Geometry

Unlike $\boldsymbol{x}^2$ which is always positive, $a^2$ can take negative values. We can classify four-vectors according to the sign of $a^2$:

**If $a^2 > 0$,** $a$ **is called *timelike,***

**If $a^2 < 0$,** $a$ **is called *spacelike,***

**If $a^2 = 0$,** $a$ **is called *lightlike*.**

Note that $\sqrt{a^2}$ has only a physical meaning for timelike vectors. It's the time measured by an observer moving along the four-vector.

### 6.2.1 Timelike

Lets extend Fig. 27 to the past. Suppose an event occurs at the origin $O$ at $\boldsymbol{x} = 0, t = 0$. Another event occurs at point $P$ at $x = 0, t < 0$. It is in the past at the same place as $O$ and hence can have some effect on $O$. Event $Q$ also occurred in the past, but at a different place. An object from $Q$ could still reach $O$ while travelling at speed below $c$. The event $Q$ could be the Circle line starting from Knightsbridge, $P$ be you arriving on the South Kensington platform and $O$ you boarding the Circle line.

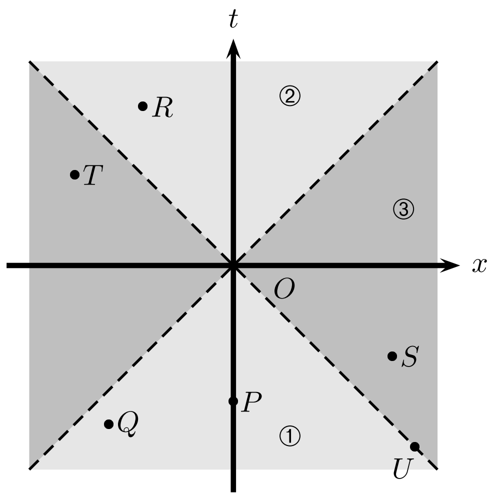


Figure 28: Space-time geometry.

Everything in zone ① below the dashed lines is in $O$'s past domain of influence. These are events that can affect $O$. Everything in zone ② above the dashed lines in the future is in $O$'s future domain of influence. These are events that could be influenced by $O$. $R$ could be you leaving the Circle line at Paddington. All points in these two domains have $ct^2 < \boldsymbol{x}^2$ and are thus *timelike*. It's the part of the universe that affects $O$ now or that $O$ is affecting.

### 6.2.2 Spacelike

Event $S$ is also in the past but too far away to affect $O$. Information about event $S$ cannot not have reached $O$. It might reach the line $x = 0$ in the future but not at time $t = 0$. For instance if Sirius exploded yesterday we wouldn't know it before 8 years. Events in these regions (zone ③) have $ct^2 > \boldsymbol{x}^2$ and are *spacelike*. The same applies to $T$ although it's in the future. $S$ and $T$ are not really different as one can always find a reference frame for which $S$ is *after* $O$ and $T$, for instance.

This is not possible for $Q$ and $O$. $Q$ will always be seen as before $O$ by any observer. Else causality would be meaningless.

### 6.2.3 Lightlike

The dashed lines are $O$'s light-cone. It's a three-dimensional surface of all events which you can see while boarding the Circle line (if in the past), or which can see $O$ in the future. Four-vectors on these lines have $ct^2 = \boldsymbol{x}^2$ and are *lightlike*.

Point $U$ for instance could be the emission of a photon from the sun arriving at South Kensington at $O$.

### 6.2.4 Time Travel

While four-vectors can be spacelike, the four-vector between two positions of the same particle cannot be. Along the trajectory of a particle in four-dimensional space-time (called "world-line") starting from the origin, the squared squared modulus $a^2$ of its position four-vector is always positive and always increasing along the worldline at speeds $u < c$.

In the own proper frame time is increasing. There's no turning back. The particle in Fig 29 crosses $x = 0$ three times, but never goes back in time and never exceeds speed of light. Although there is a minus sign in Eq. (38) the speed limit $c$ will prevent us reducing $a^2$. The distance $r$ (Eq. (36)) can become $0$ and one can come back to the initial point. But not from a trip in space-time. Time travel is excluded by special relativity.

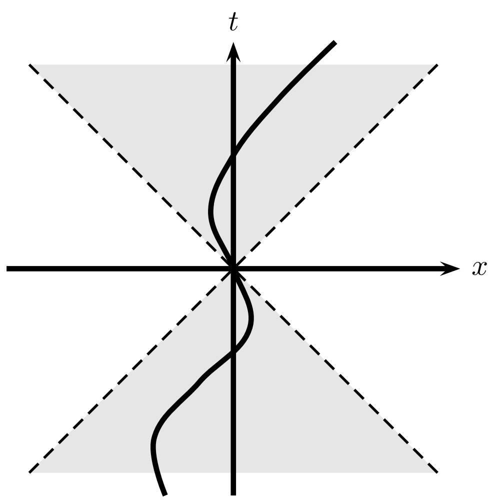


Figure 29: A "worldline": a trajectory of a particle through space-time.

## 6.3 Scalar Product

For three-vectors the squared modulus is nothing but a special case of the scalar ("dot") product.

$$\boldsymbol{x_1} \cdot \boldsymbol{x_2} = x_1 x_2 + y_1 y_2 + z_1 z_2 = |\boldsymbol{x_1}|\,|\boldsymbol{x_2}| \cos\theta$$

for any two vectors $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, where $\theta$ is the angle of the two vectors. This quantity is obviously conserved under rotations as the latter does not affect moduli nor angles.

Similarly for four-vectors we have:

**Definition — Scalar product of two four-vectors:**

$$a \cdot b \equiv c^2 t_a t_b - x_a x_b - y_a y_b - z_a z_b = c^2 t_a t_b - \boldsymbol{x_1} \cdot \boldsymbol{x_2}. \tag{40}$$

The scalar product is also *invariant* under Lorentz transformations.

The proof is in Problem 3.1.

## 6.4 Four-Momentum

Equations (32) and (33) show that $p$ transforms like $x$ and $E/c$ like $t$. All we have said for $a = (ct, \boldsymbol{x})$ is also valid for $P = (E/c, \boldsymbol{p})$.

**Definition — Energy-momentum four-vector:**

$$P \equiv \left(\frac{E}{c}, p_x, p_y, p_z\right) = \left(\frac{E}{c}, \boldsymbol{p}\right). \tag{41}$$

The full name *energy-momentum four-vector* is often shortened *four-momentum* in physicists' jargon. That's what it is called on `wikipedia`. Taylor and Wheeler [4] invent the neologism *momenergy*, which it is hardly ever used elsewhere.

We now have a new Lorentz-invariant, the scalar product of two four-momentum vectors

$$P_1 \cdot P_2 \equiv \frac{E_1 E_2}{c^2} - \boldsymbol{p_1} \cdot \boldsymbol{p_2}. \tag{42}$$

which will prove very useful for problem-solving. If we take $P = P_1 = P_2$ we get the squared modulus of a four-momentum

$$P^2 = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2 c^2, \tag{43}$$

which is as invariant as it gets.

### 6.4.1 Creation of anti-protons

Anti-protons $\overline{\mathrm{p}}$ are created in collisions of accelerated protons with a target in the reaction

$$\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{p} + \mathrm{p} + \overline{\mathrm{p}}$$

where the first proton is the incoming one and the second a proton of the target. They both also come out, but transfer some energy for the creation of a $\mathrm{p} + \overline{\mathrm{p}}$ pair. What is the minimal energy for this reaction to happen?

The first two protons have four-momenta in the laboratory frame $\mathcal{O}$

$$P_1 = \left(\frac{E}{c}, |p|, 0, 0\right) \quad \text{and} \quad P_2 = \left(\frac{mc^2}{c}, 0, 0, 0\right).$$

The incoming 4-momentum is the sum

$$P_{\text{in}} = P_1 + P_2 = \left(\frac{E + mc^2}{c}, |\boldsymbol{p}|, 0, 0\right).$$

This is equal to the outgoing 4-momentum $P_{\text{out}}$, but the latter is difficult to relate to anything useful about the four outgoing particles.

It is much easier in the centre-of-mass frame $\mathcal{O}'$. At the threshold, i.e. when the energy is just enough to produce the additional pair, the pack of three protons and the anti-proton is at rest in $\mathcal{O}'$. The total momentum in this frame is $0$ and the energy $4mc^2$

$$P'_{\text{out}} = \left(\frac{4mc^2}{c}, 0, 0, 0\right).$$

The prime reminds us that we are in a different frame. Using Eq. (43) we can write

$$\begin{aligned}
P_{\text{in}}^2 &= P'^2_{\text{out}} \\
\left(\frac{E + mc^2}{c}\right)^2 - \boldsymbol{p}^2 &= (4mc)^2 \\
\frac{E^2}{c^2} + 2Em + m^2c^2 - \frac{E}{c^2} + m^2c^2 &= 16m^2c^2 \qquad \text{(using Eq. 26)} \\
2Em &= 14m^2c^2 \\
E &= 7mc^2.
\end{aligned}$$

One needs to accelerate a proton to a kinetic energy of six proton masses to produce one anti-proton.

You can redo the calculation in the case both incoming protons are colliding head-on at the same energy and would find you need only $K = mc^2$ for each proton. In the fixed target case the energy of four proton masses is wasted in useless momentum in the fixed target experiment. This is why modern particle physics experiments all run using colliders.

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