Friday 20th May 2005

Quantum Physics Answer Sheet 3

Working with Wavefunctions

1. (i) The probability density $f(x,t)$ is proportional to $|\psi(x,t)|^2$. Hence, the probability $f(x,t)dx$ that the particle is found between x and $x+dx$ at time t is proportional to

$$\begin{aligned} |\psi(x,t)|^2\,dx &= \begin{cases} \cos^2(\pi x/d)e^{-i(\hbar\pi^2/2md^2)t}e^{i(\hbar\pi^2/2md^2)t}\,dx & |x|<d/2 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \cos^2(\pi x/d)\,dx & |x|<d/2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

which is independent of time. The particle is most likely to be found at $x=0$, where $f(x)$ is largest.

(ii) In order to normalise $\psi(x,t)$ we have to evaluate

$$N = \int_{-\infty}^{\infty}|\psi(x,t)|^2\,dx = \int_{-d/2}^{d/2}\cos^2(\pi x/d)\,dx = \frac{d}{2}\,.$$

(The integration is easy because $\cos^2\theta$ repeats every π radians and averages to $1/2$ over any whole number of repeats.)

Hence, the normalised ground-state wavefunction is:

$$\psi(x,t) = \begin{cases} \sqrt{\dfrac{2}{d}}\cos(\pi x/d)e^{-i(\hbar\pi^2/2md^2)t} & |x|<d/2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

(iii)
$$\begin{aligned} \langle x\rangle &= \int_{-d/2}^{d/2} x\,\frac{2}{d}\cos^2\left(\frac{\pi x}{d}\right)dx = 0 \qquad \text{(integrand is odd).} \\ \langle x^2\rangle &= \int_{-d/2}^{d/2} x^2\,\frac{2}{d}\cos^2\left(\frac{\pi x}{d}\right)dx \\ &= \frac{d^3}{\pi^3}\frac{2}{d}\int_{-\pi/2}^{\pi/2}\theta^2\cos^2\theta\,d\theta \qquad \text{(where } \theta=\pi x/d\text{)} \\ &= \left(\frac{1}{12}-\frac{1}{2\pi^2}\right)d^2 \qquad \text{(using integral given in question).} \end{aligned}$$

Hence

$$\Delta x = \sqrt{\langle(x-\langle x\rangle)^2\rangle} = \sqrt{\langle x^2\rangle} = d\sqrt{\frac{1}{12}-\frac{1}{2\pi^2}} \approx 0.181\,d\,.$$

2. (i) The electron probability density $f(\mathbf{r},t)$ is the square modulus of the normalised wavefunction and is proportional to the square modulus of the unnormalised wavefunction. Hence

$$f(\mathbf{r},t) \;\propto\; \left(e^{-r/a_0}e^{-iE_0t/\hbar}\right)\left(e^{-r/a_0}e^{-iE_0t/\hbar}\right)^* \;=\; e^{-2r/a_0}\;,$$

which is independent of time. The probability density is largest at $r=0$ (when the electron is right on top of the nucleus).

(ii) The normalisation condition in three dimensions is

$$\int_{\text{all space}} |\psi(\mathbf{r},t)|^2 dV \;=\; 1\;.$$

To verify that the wavefunction given in the question is normalised, we have to show that the normalisation integral,

$$N \;=\; \int_{\text{all space}} \frac{1}{\pi a_0^3} e^{-2r/a_0}\, dV\;,$$

is equal to 1. Since the integrand only depends on the length r of the vector $\mathbf{r}$, the integral can be evaluated by summing contributions from spherical shells. The contribution from the shell with inner and outer radii r and $r+dr$ is the volume of that shell, $4\pi r^2 dr$, times the value of the integrand for that shell. Hence

$$\begin{aligned} N &= \int_{r=0}^{\infty} \frac{1}{\pi a_0^3} e^{-2r/a_0} 4\pi r^2\, dr \\ &= \frac{4\pi}{\pi a_0^3}\left(\frac{a_0}{2}\right)^3 \int_{\xi=0}^{\infty} e^{-\xi}\xi^2\, d\xi \qquad \text{(where } \xi = 2r/a_0\text{)} \\ &= \frac{1}{2}\,2! \;=\; 1 \qquad \text{(using integral given in question)} \end{aligned}$$

as required.

(iii) Let $p(r)dr$ be the probability that the electron is found in the spherical shell with inner and outer radii r and $r+dr$. Then

$$p(r)\,dr \;=\; |\psi(r,t)|^2\; 4\pi r^2 dr \;=\; \frac{1}{\pi a_0^3} e^{-2r/a_0} 4\pi r^2 dr \;=\; \frac{4}{a_0^3} r^2 e^{-2r/a_0}\, dr\;.$$

The most probable distance of the electron from the nucleus is the value of r for which $p(r)$ is maximised. To find the maximum of $p(r)$, we set dp/dr equal to zero:

$$\frac{dp}{dr} \;=\; \frac{4}{a_0^3}\left(2r - \frac{2}{a_0}r^2\right) e^{-2r/a_0} \;=\; 0\;.$$

The only solution to this equation is $r = r^2/a_0$ and hence $r = a_0$. The most probable distance of the electron from the nucleus is therefore a_0. (Since $p(r)$ is a non-negative function which is zero at $r=0$, tends to zero as $r\to\infty$, and has only a single stationary point in between, there is no need to check the type of that stationary point. It can only be a maximum. Why?)

In part (i) we found that the probability density is maximised at the origin. Here we found that the most likely distance of the electron from the origin is a_0. These results are not inconsistent. The probability density $f(\mathbf{r})$ is defined as follows:

$$\text{prob. electron is found in volume element } dV \text{ at } \mathbf{r} \;=\; f(\mathbf{r})\,dV\ .$$

In other words, $f(\mathbf{r})$ is the probability per unit volume. The probability of finding the electron in the spherical shell with inner and outer radii r and $r+dr$ is the product of the probability density, $f(\mathbf{r})$, and the volume $4\pi r^2 dr$. Although the probability density peaks at the origin and decreases as r increases, the volume of a shell of fixed thickness dr increases with r. In fact, for $r < a_0$, the shell volume increases faster than the probability density decreases. The probability of finding the electron within a spherical shell of thickness dr therefore increases as r increases from 0 to a_0.

(iv) The mean distance from the nucleus is

$$\begin{aligned}
\langle r\rangle &= \int_{\text{all space}} r\,|\psi(\mathbf{r},t)|^2\,dV \;=\; \int_{r=0}^{\infty} r\,\frac{e^{-2r/a_0}}{\pi a_0^3}\,4\pi r^2 dr \\
&= \frac{4}{a_0^3}\left(\frac{a_0}{2}\right)^4 \int_0^\infty \xi^3 e^{-\xi}\,d\xi \qquad (\text{where } \xi = 2r/a_0) \\
&= \frac{a_0}{4}\,3! \qquad (\text{using integral given in question}) \\
&= \frac{3a_0}{2}\ .
\end{aligned}$$

The mean square distance from the nucleus is

$$\begin{aligned}
\langle r^2\rangle &= \int_{\text{all space}} r^2\,|\psi(\mathbf{r},t)|^2\,dV \;=\; \int_{r=0}^{\infty} r^2\,\frac{e^{-2r/a_0}}{\pi a_0^3}\,4\pi r^2 dr \\
&= \frac{4}{a_0^3}\left(\frac{a_0}{2}\right)^5 \int_0^\infty \xi^4 e^{-\xi}\,d\xi \qquad (\text{where } \xi = 2r/a_0) \\
&= \frac{a_0^2}{8}\,4! \qquad (\text{using integral given in question}) \\
&= 3a_0^2\ .
\end{aligned}$$

Hence, the root mean square distance (the square root of the mean square distance) of the electron from the nucleus is $\sqrt{3}\,a_0$.

(v) The probability of finding the electron within a sphere of radius $10^{-15}\,\text{m}$ is given exactly by the following integral:

$$\int_{r=0}^{10^{-15}\,\text{m}} |\psi(\mathbf{r},t)|^2 4\pi r^2 dr\ .$$

However, since $10^{-15}\,\text{m}$ is such a small radius, the probability density $|\psi(\mathbf{r},t)|^2$ is almost constant throughout the region of integration. A very good approximation to the probability may therefore be obtained by multiplying the probability density at the origin, $|\psi(\mathbf{0},t)|^2 = 1/\pi a_0^3$, by the volume of the tiny sphere, $\frac{4}{3}\pi(10^{-15})^3\,\text{m}^3$.

Using this method, the estimate of the probability of finding the electron on top of the nucleus is

$$\frac{\frac{4}{3}\pi\left(10^{-15}\right)^3}{\pi a_0^3} \;=\; \frac{4}{3}\frac{10^{-45}}{(0.53)^3\times 10^{-30}} \;\approx\; 9\times 10^{-15}\ .$$

Momentum Measurements

3. Since

$$e^{i\theta}=\cos\theta+i\sin\theta \qquad \text{and} \qquad e^{-i\theta}=\cos\theta-i\sin\theta\ ,$$

it follows that

$$\sin(kx)=\frac{1}{2i}\left(e^{ikx}-e^{-ikx}\right)\ .$$

Hence, in the region $x<0$, we have:

$$\psi(x,t)=\frac{-i}{2}e^{ikx-i\omega t}+\frac{i}{2}e^{-ikx-i\omega t}\ .$$

The first term is a right-going travelling wave and the second term is a left-going travelling wave.

In lectures, we considered a general superposition of travelling waves,

$$\psi(x,t)=\sum_n A(k_n)e^{i(k_nx-\omega t)}\ ,$$

and argued that $|A(k_n)|^2$ was proportional to the probability of obtaining the result $\hbar k_n$ in a measurement of momentum. The wavefunction in this question is a simple two-term example of the general form considered in lectures. The wavevectors of the two terms are k and $-k$ and their complex amplitudes are $A(k)=-i/2$ and $A(-k)=i/2$. Hence, the two possible results of a momentum measurement are $\hbar k$ and $-\hbar k$. Since the two terms have the same intensity, $|A(k)|^2=|A(-k)|^2$, the probabilities of measuring $\hbar k$ and $-\hbar k$ are both $1/2$.

Physically, the wavefunction represents a particle in a very (infinitely) spread out wavepacket of momentum $\hbar k$, which is being reflecting elastically from a potential barrier at $x=0$. Since the wavepacket is so spread out, it seems reasonable that the probability of measuring the incident momentum $\hbar k$ is equal to the probability of measuring the reflected momentum $-\hbar k$.

The Uncertainty Principle

4. The size of a nucleus is about $10^{-15}\,\mathrm{m}$ and so the position uncertainty Δx of an electron contained in a nucleus is also about $10^{-15}\,\mathrm{m}$. According to the uncertainty principle, the momentum uncertainty of such an electron satisfies

$$\Delta p \;\geq\; \frac{\hbar}{2\Delta x} \;\approx\; 5.25\times 10^{-20}\ \mathrm{kg\,m\,s^{-1}}\ .$$

Since a confined electron is not going anywhere on average, its average momentum $\langle p \rangle$ must be zero. Hence

$$(\Delta p)^2 \;=\; \langle (p - \langle p \rangle)^2 \rangle \;=\; \langle p^2 \rangle \; .$$

The kinetic energy of the electron, $\frac{1}{2} m \langle v^2 \rangle = \langle p^2 \rangle / 2m$, is therefore equal to $(\Delta p)^2/2m$:

$$KE \;=\; \frac{(\Delta p)^2}{2m} \;\geq\; \frac{(5.25 \times 10^{-20})^2}{2 \times 9.11 \times 10^{-31}} \;\approx\; 1.51 \times 10^{-9}\,\mathrm{J} \;\approx\; 9.45\,\mathrm{GeV}.$$

This shows that the kinetic energy of the confined electron must be at least 9.45 GeV, and hence that an electron of energy 1 MeV could not have been contained in the nucleus before the radioactive decay process took place.

To be careful, we should also consider the potential energy of the confined electron. If we assume that the electron is 10^{-15} m away from a nucleus with atomic number $Z = 50$, the potential energy (given by $-Ze^2/4\pi\epsilon_0 r$) is only about -0.07 GeV. This is negligible in comparison with the kinetic energy.

5. (i) Starting from the definition of the rms momentum,

$$(\Delta p)^2 \;=\; \langle (p - \langle p \rangle)^2 \rangle \; ,$$

and using the result $\langle p \rangle = 0$ given in the question, we obtain $(\Delta p)^2 = \langle p^2 \rangle$. Similarly, we can show that $(\Delta x)^2 = \langle x^2 \rangle$. The expression for the total energy,

$$\langle E \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2} s \langle x^2 \rangle \; ,$$

may therefore be rewritten as

$$\langle E \rangle \;=\; \frac{(\Delta p)^2}{2m} + \frac{1}{2} s (\Delta x)^2 \;=\; \frac{(\Delta p)^2}{2m} + \frac{1}{2} m\omega^2 (\Delta x)^2 \; ,$$

where the last step followed because $\omega = \sqrt{s/m}$.

(ii) The uncertainty principle states that $\Delta x \Delta p \geq \hbar/2$. If we use this to eliminate Δp from the expression for $\langle E \rangle$ we obtain

$$\langle E \rangle \;\geq\; \frac{1}{2m} \frac{\hbar^2}{4(\Delta x)^2} + \frac{1}{2} m\omega^2 (\Delta x)^2 \;=\; \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2} m\omega^2 (\Delta x)^2 \; .$$

The value of Δx that makes the right-hand side as small as possible (and hence imposes the weakest possible condition on $\langle E \rangle$) may be found by differentiating with respect to Δx and setting the result equal to zero:

$$\frac{d}{d(\Delta x)} \left[\frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2} m\omega^2 (\Delta x)^2 \right] \;=\; \frac{-\hbar^2}{4m(\Delta x)^3} + m\omega^2 \Delta x \;=\; 0 \; .$$

Solving this equation gives $\Delta x = \sqrt{\hbar/2m\omega}$. Substituting back into the inequality for $\langle E \rangle$ gives

$$\langle E \rangle \;\geq\; \frac{\hbar^2}{8m(\hbar/2m\omega)} + \frac{1}{2} m\omega^2 \left(\frac{\hbar}{2m\omega} \right) \;=\; \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega \; ,$$

and hence

$$\langle E \rangle \;\geq\; \frac{1}{2}\hbar\omega \,,$$

as required.

6. The energy-time uncertainty principle,

$$\Delta E \Delta t \;\geq\; \frac{\hbar}{2} \,,$$

relates the lifetime Δt of a state to its energy uncertainty ΔE. For $\Delta t = 2.6 \times 10^{-10}\,\mathrm{s}$ we obtain

$$\Delta E \;\geq\; \frac{1.05 \times 10^{-34}}{2 \times 2.6 \times 10^{-10}} \;\approx\; 2.02 \times 10^{-25}\,\mathrm{J} \;\approx\; 1.26 \times 10^{-6}\,\mathrm{eV}.$$

Hence, the minimum energy uncertainty of the emitted photons is $1.26 \times 10^{-6}\,\mathrm{eV}$.

7. In order for a virtual particle of energy mc^2 to pop out of the vacuum, the energy uncertainty ΔE must be (at least) mc^2. According to the energy-time uncertainty principle in the form $\Delta E \Delta t \sim \hbar$, the lifetime of such a particle is $\Delta t \sim \hbar/\Delta E \sim \hbar/mc^2$. The speed of the particle must be less than the speed of light, and hence the distance it travels during its lifetime must be less than $c\Delta t \sim \hbar/mc$.

Equating this distance to the range of the nuclear force gives

$$\frac{\hbar}{mc} \;\approx\; 1.4 \times 10^{-15}\,\mathrm{m} \,,$$

and hence

$$mc^2 \;\approx\; \frac{\hbar c}{1.4 \times 10^{-15}} \;=\; \frac{1.05 \times 10^{-34} \times 3.00 \times 10^{8}}{1.4 \times 10^{-15}} \;=\; 2.25 \times 10^{-11}\,\mathrm{J} \,.$$

In eV, this becomes

$$mc^2 \;\approx\; \frac{2.25 \times 10^{-11}}{1.60 \times 10^{-19}} \;\approx\; 141 \times 10^{6}\,\mathrm{eV} \;=\; 141\,\mathrm{MeV} \,,$$

or, equivalently, $m \approx 141\,\mathrm{MeV}/c^2$.

The Schrödinger Equation

8. The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) \;=\; E\psi(x) \,.$$

If the wavefunction

$$\psi_n(x) \;=\; \begin{cases} \sqrt{\frac{2}{d}}\sin\left(\frac{n\pi x}{d}\right) & 0 < x < d \\ 0 & \text{otherwise} \end{cases}$$

is a normalised energy eigenfunction, the following conditions hold:

(a) $\psi_n(x)$ satisfies the boundary conditions: $\psi_n(0) = \psi_n(d) = 0$.

(b) $\psi_n(x)$ satisfies the time-independent Schrödinger equation for $0 < x < d$.

(c) $\psi_n(x)$ is normalised.

Consider these conditions one by one:

(a) By inspection, $\psi_n(x)$ satisfies the boundary conditions for $n = 1, 2, \ldots$.

(b) Substitute $\psi_n(x)$ into the left-hand side of the Schrödinger equation for $0 < x < d$:

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} + V(x)\psi_n(x) &= -\frac{\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} \qquad [V(x) = 0 \text{ for } 0 < x < d] \\ &= -\frac{\hbar^2}{2m}\sqrt{\frac{2}{d}}\frac{d^2}{dx^2}\sin\left(\frac{n\pi x}{d}\right) \\ &= \frac{\hbar^2}{2m}\left(\frac{n\pi}{d}\right)^2\sqrt{\frac{2}{d}}\sin\left(\frac{n\pi x}{d}\right) \\ &= E_n\psi_n(x)\,, \quad \text{where} \quad E_n = \frac{\hbar^2 n^2\pi^2}{2md^2}\,. \end{aligned}$$

Hence, $\psi_n(x)$ satisfies the time-independent Schrödinger equation in the region $0 < x < d$. The corresponding energy eigenvalue is $E_n = \hbar^2 n^2\pi^2/2md^2$.

(c) If $\psi_n(x)$ is normalised, the integral

$$N = \int_0^d |\psi_n(x)|^2\,dx$$

must be equal to 1. Check this by evaluating the integral:

$$N = \frac{2}{d}\int_0^d \sin^2\left(\frac{n\pi x}{d}\right)dx = \frac{2}{d}\times\frac{d}{2} = 1\,.$$

Hence, $\psi_n(x)$ is normalised.

9. Using the result of Q9, the energy eigenvalues are:

$$E_n = \frac{\hbar^2 n^2\pi^2}{2md^2} \qquad n = 1, 2, \ldots\,.$$

The energy emitted as the nucleon falls from the $n = 2$ level to the $n = 1$ level is

$$\begin{aligned} E_2 - E_1 &= \frac{3\hbar^2\pi^2}{2md^2} \approx \frac{3\times(1.05\times10^{-34})^2\times\pi^2}{2\times1.67\times10^{-27}\times(10^{-15})^2} \\ &\approx 9.8\times10^{-11}\,\text{J} \approx 610\,\text{MeV}\,. \end{aligned}$$

This is a sensible number. The energy released in fission is about 200 MeV per nucleus.

10. The classical angular frequency of vibration is $\omega_{\text{vib}} = \sqrt{s/m_{\text{reduced}}}$, where

$$m_{\text{reduced}} = \left(\frac{1}{m_{\text{C}}} + \frac{1}{m_{\text{O}}}\right)^{-1} = \left(\frac{1}{12} + \frac{1}{16}\right)^{-1} \approx 6.86 \text{ atomic mass units.}$$

Hence

$$\omega_{\rm vib} \;=\; \sqrt{\frac{1857}{6.86\times 1.66\times 10^{-27}}} \;\approx\; 4.04\times 10^{14}\ \text{Radians}\,\text{s}^{-1}\ .$$

(i) The energy of the photon emitted when a CO molecule makes a transition between neighbouring vibrational states is the difference between the energies of those states. The energy levels of a QM simple harmonic oscillator are $(n+\frac{1}{2})\hbar\omega_{\rm vib}$, where $n = 0, 1, 2, \ldots$. Hence, the energy E of the emitted photon is $\hbar\omega_{\rm vib}$. Since the energy E and angular frequency ω of the photon are related via $E = \hbar\omega$, it follows that $\omega = \omega_{\rm vib}$. The frequency of the emitted photon is therefore the same as the classical vibrational frequency of the molecule, which makes sense. The photon wavelength is

$$\lambda \;=\; \frac{c}{\nu} \;=\; \frac{c}{\nu_{\rm vib}} \;=\; \frac{2\pi c}{\omega_{\rm vib}} \;\approx\; 4.67\times 10^{-6}\,\text{m}\ .$$

(ii) The vibrational zero-point energy of a CO molecule is

$$\frac{1}{2}\hbar\omega_{\rm vib} \;\approx\; 2.12\times 10^{-20}\,\text{J} \;\approx\; 0.13\,\text{eV}\ .$$

11. Substitute the trial solution $\psi(x) = e^{-\alpha x^2}$ into the left-hand side of the Schrödinger equation:

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}sx^2\psi(x) &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}e^{-\alpha x^2} + \frac{1}{2}sx^2e^{-\alpha x^2} \\
&= -\frac{\hbar^2}{2m}\frac{d}{dx}\left(-2\alpha x e^{-\alpha x^2}\right) + \frac{1}{2}sx^2e^{-\alpha x^2} \\
&= \frac{\hbar^2\alpha}{m}\frac{d}{dx}\left(x e^{-\alpha x^2}\right) + \frac{1}{2}sx^2e^{-\alpha x^2} \\
&= \frac{\hbar^2\alpha}{m}\left(e^{-\alpha x^2} - 2\alpha x^2 e^{-\alpha x^2}\right) + \frac{1}{2}sx^2e^{-\alpha x^2} \\
&= \frac{\hbar^2\alpha}{m}e^{-\alpha x^2} + \left(\frac{1}{2}s - \frac{2\hbar^2\alpha^2}{m}\right)x^2e^{-\alpha x^2}\ .
\end{aligned}$$

If $\psi(x)$ is to be a solution of the Schrödinger equation, this must equal a constant, E, times $e^{-\alpha x^2}$. The value of α must therefore be chosen to ensure that the coefficient of the $x^2e^{-\alpha x^2}$ term is zero. Hence

$$\alpha^2 \;=\; \frac{ms}{4\hbar^2} \quad\Rightarrow\quad \alpha \;=\; \frac{\sqrt{sm}}{2\hbar}\ .$$

If this condition is met, $\psi(x)$ is an energy eigenfunction with energy eigenvalue

$$E \;=\; \frac{\hbar^2\alpha}{m} \;=\; \frac{\hbar^2}{m}\frac{\sqrt{sm}}{2\hbar} \;=\; \frac{1}{2}\hbar\sqrt{\frac{s}{m}} \;=\; \frac{1}{2}\hbar\omega\ ,$$

where $\omega = \sqrt{s/m}$ is the classical angular frequency of the oscillator.

Q6 showed that the energy of a quantum mechanical simple harmonic oscillator must be $\geq \frac{1}{2}\hbar\omega$, and so the eigenfunction found here must be the (unnormalised) ground state.

12. Inside the barrier, the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V\psi(x) \;=\; E\psi(x) \;,$$

with $V > E$. A simple rearrangement gives

$$\frac{d^2\psi(x)}{dx^2} \;=\; \frac{2m(V-E)}{\hbar^2}\psi(x) \;=\; \gamma^2\psi(x) \;,$$

where

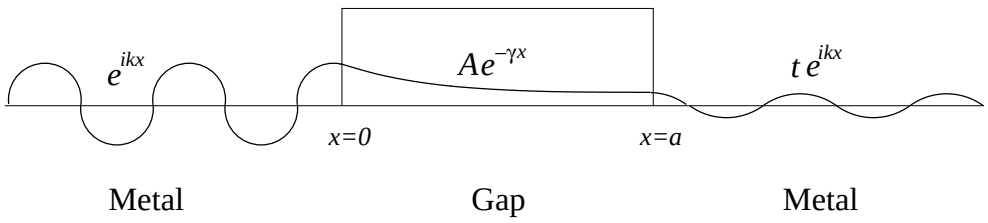
$$\gamma = \sqrt{\frac{2m(V-E)}{\hbar^2}}$$

is real because $V - E$ is greater than zero. By inspection, the two independent solutions of this second-order differential equation are $e^{\gamma x}$ and $e^{-\gamma x}$, and so the general solution is

$$\psi(x) = Ae^{-\gamma x} + Be^{\gamma x} \;,$$

where A and B are arbitrary constants to be determined from the boundary conditions.

Assume now that the barrier stretches from $x = 0$ to $x = a$ and that the particles are incident from the left. We discard the $e^{\gamma x}$ solution on physical grounds: it would make little sense if the wavefunction, and hence the probability density, increased with distance into the barrier. (In fact, as you will find out next year, the exact wavefunction inside a barrier of finite width does include a small amount of the $e^{\gamma x}$ solution. However, as long as the barrier is wide and the tunnelling probability low, the value of B is so small that this contribution can be ignored.) Inside the barrier, then, the wavefunction $\psi(x)$ is equal to $Ae^{-\gamma x}$, as shown in the figure.



It follows that

$$\frac{|\psi(x=a)|^2}{|\psi(x=0)|^2} \;\approx\; \left(e^{-\gamma a}\right)^2 \;=\; e^{-2\gamma a} \;.$$

Since $|\psi(x)|^2$ is proportional to the probability density at position x, this equation shows that the probability density at the right-hand edge of the barrier is $e^{-2\gamma a}$ (which is normally a very small number) times the probability density at the left-hand edge. The tunnelling probability is therefore equal to $e^{-2\gamma a}$.

The work function W is the minimum energy required to remove an electron from a piece of metal. If additional energy W is supplied to a metallic electron with energy E, that electron is only just able to escape from the surface. This implies that the potential energy V of the electron outside the metal must be $E + W$. If no extra energy is supplied (so

that the electron from the metal still has energy E), the Schrödinger equation in the gap region takes the form,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + (E+W)\psi(x) \;=\; E\psi(x)\;,$$

and hence $\gamma = \sqrt{2m(E+W-E)/\hbar^2} = \sqrt{2mW/\hbar^2}$.

In the case when $a = 10^{-6}\,\mathrm{m}$ and $W = 5\,\mathrm{eV}$, we obtain

$$\gamma \;=\; \frac{\sqrt{2\times 9.11\times 10^{-31}\times 5\times 1.60\times 10^{-19}}}{1.05\times 10^{-34}} \;\approx\; 1.15\times 10^{10}\,\mathrm{m}^{-1}\;.$$

The tunnelling probability is therefore

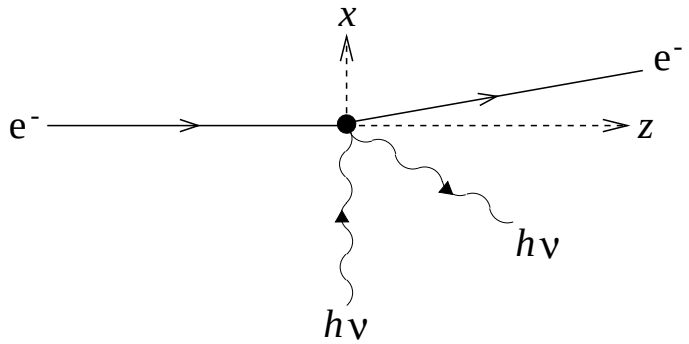
$$e^{-2\gamma a} \;\approx\; e^{-23{,}000} \;=\; 10^{-23{,}000\times\log_{10} e} \;\approx 10^{-10{,}000}\;.$$

This is *very* small! Electron tunnelling is important in scanning tunnelling microscopes, where the gap is comparable to the size of an atom ($10^{-10}\,\mathrm{m}$), but not for the much larger gap considered here.

Extra Questions for Enthusiasts

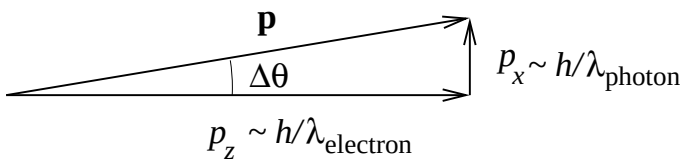
13. (i) In order to tell which slit the electron went through, the optical instrument that detects the scattered photons must be able to resolve features of size d. Since the resolution cannot be much better than the wavelength of the light used, $\lambda_{\mathrm{photon}}$ must be less than or approximately equal to d.

(ii) The figure shows a photon moving in the x direction scattering from an electron moving in the z direction.



The x momentum transferred to the electron may take any value from 0 (if the photon is not deflected at all) to $2h/\lambda_{\mathrm{photon}}$ (if the photon scatters back towards its source), but the typical value is around $h/\lambda_{\mathrm{photon}}$. The initial z momentum of the electron is $p_z = h/\lambda_{\mathrm{electron}}$. The value of p_z changes slightly when the photon scatters, but if the change in electron direction is small the change in p_z will also be small. Hence, even after the photon has been scattered, $p_z \approx h/\lambda_{\mathrm{electron}}$.

After the scattering event, the electron's momentum vector looks something like this:



If the scattering angle $\Delta\theta$ is small, so that $\tan(\Delta\theta) \approx \Delta\theta$, it follows that

$$\Delta\theta \approx \tan(\Delta\theta) = \frac{p_x}{p_z} \sim \frac{\lambda_{\text{electron}}}{\lambda_{\text{photon}}} .$$

(iii) The angular spacing between adjacent diffraction maxima or minima is $\lambda_{\text{electron}}/d$. Since $\lambda_{\text{photon}} \leq d$ (see part (i)), the angular spacing must be $\leq \lambda_{\text{electron}}/\lambda_{\text{photon}}$. But this is exactly equal to $\Delta\theta$, the uncertainty in the direction of the electron caused by scattering the photon (part (ii)). Hence, the scattering of the photon is sufficient to smear out the diffraction pattern completely.

14. The integral we have to evaluate is

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_n^*(x)\psi_m(x)\,dx &= \frac{2}{d}\int_0^d \sin\left(\frac{n\pi x}{d}\right)\sin\left(\frac{m\pi x}{d}\right)dx \\ &= \frac{2}{d}\,\frac{d}{\pi}\int_0^\pi \sin(n\theta)\sin(m\theta)\,d\theta \qquad (\text{where } \theta = \pi x/d) \\ &= \frac{1}{\pi}\int_0^\pi [\cos((n-m)\theta) - \cos((n+m)\theta)]\,d\theta ,\end{aligned}$$

where the last step used the expression for $\sin(n\theta)\sin(m\theta)$ given in the question. For any non-zero integer j, the integral

$$\int_0^\pi \cos(j\theta)\,d\theta = \left[\frac{1}{j}\sin(j\theta)\right]_0^\pi = 0 .$$

Hence, if n and m are unequal positive integers,

$$\int_{-\infty}^{\infty} \psi_n^*(x)\psi_m(x)\,dx = 0 .$$

If $n = m$, so that $\cos((n-m)\theta) = 1$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_n^*(x)\psi_m(x)\,dx &= \int_{-\infty}^{\infty} \psi_n^*(x)\psi_n(x)\,dx \\ &= \frac{1}{\pi}\int_0^\pi [1-\cos(2n\theta)]d\theta = \frac{1}{\pi}\int_0^\pi d\theta = 1 ,\end{aligned}$$

demonstrating that $\psi_n(x)$ is normalised.

15. If $\phi(x) = \sum_{n=1}^{\infty} c_n\psi_n(x)$ is normalised, then

$$\int_{-\infty}^{\infty} \phi^*(x)\phi(x)\,dx = 1 .$$

Substituting the expansion of $\phi(x)$ into the normalisation integral gives

$$\begin{aligned}\int_{-\infty}^{\infty} \phi^*(x)\phi(x)\,dx &= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} c_n^* \psi_n^*(x) \sum_{m=1}^{\infty} c_m \psi_m(x)\,dx \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_n^* c_m \int_{-\infty}^{\infty} \psi_n^*(x)\psi_m(x)\,dx\ .\end{aligned}$$

The integral in this expression is equal to 1 if $n = m$ and to 0 otherwise. Hence, all terms with $n \neq m$ vanish and only the $n = m$ terms are left:

$$\int_{-\infty}^{\infty} \phi^*(x)\phi(x)\,dx = \sum_{n=1}^{\infty} c_n^* c_n \int_{-\infty}^{\infty} \psi_n^*(x)\psi_n(x)\,dx = \sum_{n=1}^{\infty} c_n^* c_n\ .$$

This shows that $\phi(x)$ is normalised if and only if

$$\sum_{n=1}^{\infty} c_n^* c_n = 1\ .$$

Since $c_n^* c_n$ is greater than or equal to zero, and since the sum of $c_n^* c_n$ over all n gives 1, it is reasonable (and right) to guess that $c_n^* c_n$ is the probability that a measurement of the energy gives the result E_n (where E_n is the energy eigenvalue corresponding to the energy eigenfunction $\psi_n(x)$).