Friday 13th May 2005

Quantum Physics Problem Sheet 3

Note that these questions are ordered by topic, not by difficulty. Do not get disheartened if you find some of the questions near the beginning of the sheet hard.

Working with Wavefunctions

1. A particle of mass m is confined within a potential well of the form:

$$V(x) \quad = \quad \begin{cases} 0 & |x| < d/2 \ , \\ \infty & \text{otherwise} \ . \end{cases}$$

   The (unnormalised) ground-state wavefunction is

$$\psi(x,t) \quad = \quad \begin{cases} \cos\left(\pi x/d\right) e^{-i\left(\hbar\pi^2/2md^2\right)t} & |x| < d/2 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

   (i) A measurement is made of the particle position. Show that the probability that the measured value lies between x and $x+dx$ is independent of time. Where is the partice most likely to be found?

   (ii) Normalise $\psi(x,t)$.

   (iii) Find $\langle x \rangle$ and $\langle x^2 \rangle$. Hence obtain the rms width Δx of the probability density function.

$$\left[\text{The following result may be used without proof:} \quad \int_{-\pi/2}^{\pi/2} \theta^2 \cos^2\theta \; d\theta \quad = \quad \frac{\pi}{4}\left(\frac{\pi^2}{6} - 1\right)\right]$$

2. The normalised wavefunction $\psi(\mathbf{r},t)$ of a particle moving in three dimensions has the following probability interpretation:

$$|\psi(\mathbf{r},t)|^2 \, dV \quad = \quad \begin{cases} \text{probability that a measurement of the position of the} \\ \text{particle yields a value in the volume element } dV \text{ at } \mathbf{r}. \end{cases}$$

   The (unnormalised) ground-state wavefunction of the electron in a Hydrogen atom is $e^{-r/a_0} e^{-iE_0 t/\hbar}$, where $a_0 \approx 0.53\,\text{Å}$ is the Bohr radius and $E_0 \approx -13.6\,\text{eV}$ is the ground-state energy.

   (i) Show that the electron probability density is independent of time. Where is the probability density largest?

(ii) Write down the normalisation condition that must be satisfied by any wavefunction satisfying the probabiliy interpretation described above. Show that the normalised ground-state wavefunction is

$$\psi(\mathbf{r},t) \;=\; \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} e^{-iE_0 t/\hbar} \;.$$

(Hint: you will have to evaluate the integral $\int_0^\infty e^{-2r/a_0} 4\pi r^2 dr$. You may assume that $\int_0^\infty \xi^n e^{-\xi} d\xi = n!$)

(iii) A measurement is made of the electron position $\mathbf{r}$. Write down an expression for the probability that the electron is found in the spherical shell with inner and outer radii r and $r + dr$. Hence show that the most probable distance of the electron from the nucleus is a_0. Is this result consistent with the position of maximum probability density found in part (i)?

(iv) Show that the mean distance of the electron from the nucleus is $3a_0/2$ and that the rms distance is $\sqrt{3}\,a_0$.

(v) Show that the probability of finding the electron on top of the nucleus, which you may assume has radius 10^{-15} m, is approximately 9×10^{-15}. (Hint: no integration is required for this part.)

Momentum Measurements

3. The (unnormalised) wavefunction of a quantum mechanical particle that is being reflected from an infinitely high potential barrier at $x = 0$ takes the form:

$$\psi(x,t) \;=\; \begin{cases} \sin(kx)e^{-i\omega t} & x < 0 \;, \\ 0 & \text{otherwise} \;. \end{cases}$$

where $\omega = \hbar k^2/2m$. In the region $x < 0$, show that $\psi(x,t)$ may be written as a sum of right- and left-going complex travelling waves.

The momentum of the particle is measured. What values might be obtained and what are their relative probabilities?

The Uncertainty Principle

4. When undergoing radioactive decay, nuclei often emit electrons with energies between 1 and 10 MeV. Use the position-momentum uncertainty principle to show that an electron of energy 1 MeV could not have been contained in the nucleus before the decay.

5. Just as in the position and momentum examples considered in lectures, measurements of the energies of identical quantum mechanical systems do not always yield identical results. QM is able to predict the possible results of a measurement and their probabilities, but not the actual result.

(i) Consider a QM simple harmonic oscillator of mass m, spring constant s and natural frequency $\omega = \sqrt{s/m}$. Assuming that the average displacement $\langle x \rangle$ and average momentum $\langle p \rangle$ are both zero, show that the mean energy,

$$\langle E \rangle \quad = \quad \langle KE \rangle + \langle PE \rangle \quad = \quad \frac{\langle p^2 \rangle}{2m} + \frac{1}{2} s \langle x^2 \rangle \ ,$$

may be expressed in the form:

$$\langle E \rangle \quad = \quad \frac{(\Delta p)^2}{2m} + \frac{1}{2} m\omega^2 (\Delta x)^2 \ ,$$

where Δx is the rms displacement and Δp is the rms momentum.

(ii) Use the uncertainty principle to show that the ground-state energy of the oscillator must be greater than or equal to $\frac{1}{2}\hbar\omega$. (In fact, the ground-state energy is *exactly* $\frac{1}{2}\hbar\omega$.)

6. When a short-lived excited state of an atom or molecule decays to the ground state, the wavelength of the photon emitted is uncertain. What is the minimum energy uncertainty (often known as the linewidth), measured in eV, of photons emitted from a state of lifetime 2.6×10^{-10} s.

7. In 1935, Yukawa proposed that the nuclear force arises through the emission of an unknown virtual particle, a pion, by one of the nucleons and its absorption by the other.

   Assuming that $\Delta E \Delta t \sim \hbar$, relate the lifetime of the virtual pion to its mass m. Hence show that the distance travelled by the pion during its lifetime is unlikely to exceed $\hbar/mc$. Given that the range of the nuclear force is approximately 1.4×10^{-15} m, estimate m in MeV/c^2. (When the pion was finally discovered, its mass was found to be $\sim$140 MeV/c^2. The high accuracy of the estimate obtained in this question is of course fortuitous.)

The Schrödinger Equation

8. A particle of mass m is confined within a potential well of the form:

$$V(x) \quad = \quad \begin{cases} 0 & 0 < x < d \\ \infty & \text{otherwise} \end{cases}$$

   Write down the time-independent Schrödinger equation for this system. Verify that the wavefunctions

$$\psi_n(x) \quad = \quad \begin{cases} \sqrt{\dfrac{2}{d}} \sin\left(n\pi x/d\right) & 0 < x < d \\ 0 & \text{otherwise} \end{cases}$$

   where $n = 1, 2, \ldots$, are normalised energy eigenfunctions. Find the corresponding energy eigenvalues.

9. Assuming that a nucleus can be modelled as an infinite potential well of width $d = 10^{-15}$ m, estimate the energy a nucleon (mass 1.67×10^{-27} kg) emits as it falls from the $n = 2$ to the $n = 1$ level. Is this a sensible number?

10. The bond in a carbon monoxide ($^{12}C^{16}O$) molecule acts like a spring with spring constant $s = 1857\,\mathrm{Nm}^{-1}$. Calculate the angular frequency of vibration according to classical physics. (Hint: use the reduced mass.) Hence work out:

    (i) The wavelength of the photons emitted when a CO molecule makes a transition between neighbouring vibrational states.

    (ii) The vibrational zero-point energy of a CO molecule.

11. The time-independent Schrödinger equation for a simple harmonic oscillator of mass m and spring constant s is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}sx^2\psi(x) = E\psi(x) \; .$$

    Verify by subsitution that the function

$$\psi(x) = e^{-\alpha x^2}$$

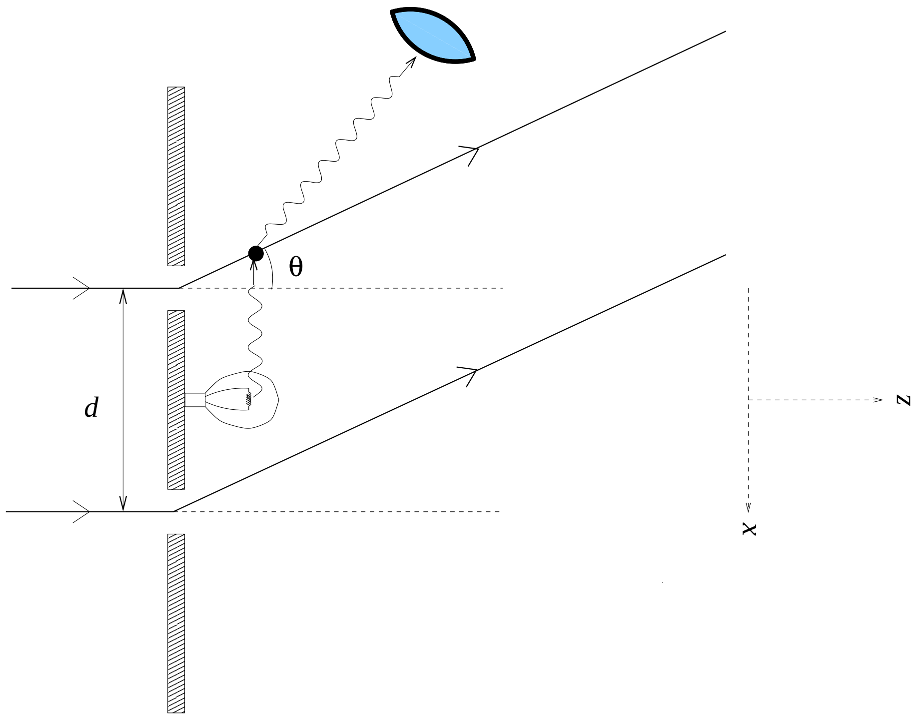
    satisfies this equation if α is chosen correctly. Show that the corresponding eigenvalue is $\frac{1}{2}\hbar\omega$, where ω is the classical angular frequency of the oscillator.

12. An electron of energy E encounters a potential barrier of height $V > E$ and width a. Show that the probability that the electron tunnels across the barrier is approximately $\exp(-2\gamma a)$, where $\gamma = \sqrt{2m(V-E)/\hbar^2}$. Estimate the probability of tunnelling across a 10^{-6} m gap between two slabs of metal of work function 5 eV.

Extra Questions for Enthusiasts (not examinable)

13. The probability density of the positions at which the electrons in a two-slit experiment hit the screen is the square modulus of a complex wave that travels through both slits. Which slit does the electron itself go through? As I tried to explain in lectures, this is not a question that quantum mechanics can answer. The point-like "particle" that we associate with the electron is simply a convenient abstraction or mental picture that we use to help interpret the results of certain types of measurement — clicks of an electron counter or flashes of light originating from specific points in space. QM tells us almost nothing about what "really" produces these experimental results, and does not allow us to ascribe a position to the particle abstraction between measurements.

    To help make these ideas more concrete, consider the thought experiment illustrated below.



When the light bulb is switched off, the electron wave passing through the two slits interferes constructively at angles θ satisfying:

$$d \sin\theta = n\lambda_{\text{electron}} \qquad (n = 0, 1, 2, \ldots).$$

If θ is small, this translates to $\theta \approx n\lambda_{\text{electron}}/d$.

Now imagine switching on the light bulb (which is shaded to ensure that it only sends light along the plane of the barrier containing the slits). If the light is bright enough, most of the electrons emerging from the slits will scatter one or more photons. By imaging the flashes of light corresponding to the scattered photons, we can find out where they came from and hence measure the positions of the electrons when they are just behind the slits.

What are the results of this experiment? As you might expect, we see flashes of light originating from both slits, but never from both at the same time. Every time we detect an electron, it is either behind one slit or behind the other. However, when we look at the screen, we find that the two-slit diffraction pattern has disappeared! The measurement of the electron position just behind the slits has somehow destroyed the interference pattern. We can have it one way (interference pattern but no measurement of electron path) or the other (measurement of electron path but no interference pattern), but not both.

(i) In order for this experiment to work, the photon wavelength must be less than or approximately equal to d. Why?

(ii) Assuming that the x component of the momentum of the electron is small (i.e., assuming that θ is small), show that the impact of a single photon of wavelength λ_{photon} deflects the electron by an angle $\Delta\theta$ of order $\lambda_{\text{electron}}/\lambda_{\text{photon}}$.

(iii) Explain why the measurement of the electron position wipes out the electron diffraction pattern.

14. In next year's QM course, you will prove that different energy eigenfunctions of the same physical system are "orthogonal" to each other, in the sense that

$$\int_{-\infty}^{\infty} \psi_n^*(x)\psi_m(x)dx = 0 \qquad n \neq m.$$

Verify this result for the energy eigenfunctions given in Q9. (Since these functions are real there is no need to worry about the complex conjugation.)

The following result may be used without proof:

$$\sin(n\theta)\sin(m\theta) = \frac{1}{2}\left[\cos((n-m)\theta) - \cos((n+m)\theta)\right]$$

15. Show that the function

$$\phi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

is normalised if and only if the (possibly complex) coefficients c_n satisfy $\sum_{n=1}^{\infty} |c_n|^2 = 1$. (Hint: use the orthogonality relation from Q15 and remember that $\psi_n(x)$ is normalised.) Hazard a guess at the probability interpretation of $|c_n|^2$.

Physical Constants

$$\begin{aligned}
m_{\text{electron}} &\approx 9.11 \times 10^{-31}\,\text{J} \approx 511\,\text{keV}/c^2 \\
\text{atomic mass unit} &\approx 1.66 \times 10^{-27}\,\text{kg} \\
h &\approx 6.63 \times 10^{-34}\,\text{Js} \\
\hbar &\approx 1.05 \times 10^{-34}\,\text{Js} \\
c &\approx 3.00 \times 10^{8}\,\text{ms}^{-1} \\
e &\approx 1.60 \times 10^{-19}\,\text{C}
\end{aligned}$$

Numerical Answers

6. 1.26×10^{-6} eV.
7. $141\,\text{MeV}/c^2$.
9. 610 MeV.
10. (i) 4.67×10^{-6} m; 0.13 eV.
12. $\sim e^{-23,000} \sim 10^{-10,000}$.