Tuesday 3rd May 2005

Quantum Physics Answer Sheet 2

Compton Scattering

1. (i) The wavelength of the incident photons is

$$\lambda = \frac{hc}{E} \approx \frac{6.63\times10^{-34}\times3.00\times10^{8}}{20\times10^{3}\times1.60\times10^{-19}} \approx 6.22\times10^{-11}\,\mathrm{m}\ .$$

The change in wavelength is given by the Compton formula with $\theta = 60^o$:

$$\lambda' - \lambda = \frac{h}{mc}(1-\cos\theta) \approx \frac{6.63\times10^{-34}\times0.5}{9.11\times10^{-31}\times3.00\times10^{8}} \approx 1.21\times10^{-12}\,\mathrm{m}\ .$$

Combining the values of λ and $\lambda' - \lambda$ gives the wavelength of the scattered photons:

$$\lambda' \approx 6.34\times10^{-11}\,\mathrm{m}\ .$$

(ii) The energy lost by a photon as it scatters is:

$$\begin{aligned}
E - E' &= hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)\\
&= 6.63\times10^{-34}\times3.00\times10^{8}\left(\frac{1}{6.22\times10^{-11}} - \frac{1}{6.34\times10^{-11}}\right)\\
&\approx 6.05\times10^{-17}\,\mathrm{J}\\
&\approx 378\,\mathrm{eV}\ .
\end{aligned}$$

Warning: this answer is the difference of two much larger numbers (the incoming and outgoing photon energies) and is subject to considerable rounding error. For example, if you store intermediate values such as λ and λ' to full calculator precision, the final result changes by several eV. Short of using more accurate values for the fundamental constants, there is little that can be done about this.

All this energy is transferred to the electron as recoil energy. The work function of a typical solid is only 5 or 10 eV, so some of the recoiling electrons will certainly escape from the metal.

The largest change in wavelength would be obtained when $\theta = 180^0$, in which case $\lambda' - \lambda = 2h/mc \approx 4.85\times10^{-12}\,\mathrm{m}$. The maximum possible wavelength of the scattered photon (assuming only one scattering) is $6.22\times10^{-11} + 4.85\times10^{-12} \approx 6.71\times10^{-11}\,\mathrm{m}$.

2. (i) The initial photon wavelength is

$$\lambda_{\mathrm{init}} = \frac{hc}{E} \approx \frac{6.63\times10^{-34}\times3.00\times10^{8}}{10^{6}\times1.60\times10^{-19}} \approx 1.24\times10^{-12}\,\mathrm{m}\ .$$

The final photon wavelength after 10^{26} Compton scattering events is 500 nm. If we assume that each scattering event increases the photon wavelength by the same amount $\Delta\lambda$, we obtain

$$10^{26}\Delta\lambda \approx (500 \times 10^{-9} - 1.24 \times 10^{-12})\,\mathrm{m}\ ,$$

and hence

$$\Delta\lambda \approx 5 \times 10^{-33}\,\mathrm{m}\ .$$

(ii) The Compton formula says that

$$\Delta\lambda = \frac{h}{mc}(1 - \cos\theta)\ .$$

Since $\Delta\lambda$ ($\approx 5 \times 10^{-33}$ m) $\ll h/mc$ ($\approx 2.43 \times 10^{-12}$ m), the average scattering angle θ must be very small (so that $\cos\theta$ is very close to 1). We can therefore make the approximation $\cos\theta \approx 1 - \theta^2/2$ to obtain $\Delta\lambda \approx h\theta^2/2mc$, and hence

$$\begin{aligned}
\theta &\approx \sqrt{\frac{2mc\Delta\lambda}{h}} \\
&\approx \sqrt{\frac{2 \times 9.11 \times 10^{-31} \times 3.00 \times 10^{8} \times 5 \times 10^{-33}}{6.63 \times 10^{-34}}} \\
&\approx 6.42 \times 10^{-11}\ \text{radians} \\
&\approx 3.68 \times 10^{-9}\ \text{degrees}\ .
\end{aligned}$$

(iii) In 10^6 years, a photon travels a distance:

$$\begin{aligned}
d = ct &= 3.00 \times 10^{8} \times 60 \times 60 \times 24 \times 365 \times 10^{6} \\
&\approx 9.46 \times 10^{21}\,\mathrm{m}\ .
\end{aligned}$$

During this time, it scatters 10^{26} times. Hence, the average distance travelled by a photon between scattering events is $9.46 \times 10^{21}/10^{26} \approx 9.46 \times 10^{-5}$ m or roughly 0.1 mm.

De Broglie Waves

3. In order to use neutron diffraction to study atomic positions and atomic-scale magnetic fields, the neutron De Broglie wavelength must be comparable to the size of an atom:

$$\lambda \approx 10^{-10}\,\mathrm{m}\ .$$

The kinetic energy of the neutron is thus:

$$\begin{aligned}
\frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} &\approx \frac{(6.63 \times 10^{-34})^2}{2 \times 1.67 \times 10^{-27} \times 10^{-20}} \\
&\approx 1.32 \times 10^{-20}\,\mathrm{J} \approx 0.083\,\mathrm{eV}\ .
\end{aligned}$$

This is the same as the average energy $3k_BT/2$ of a neutron in thermal equilibrium at temperature

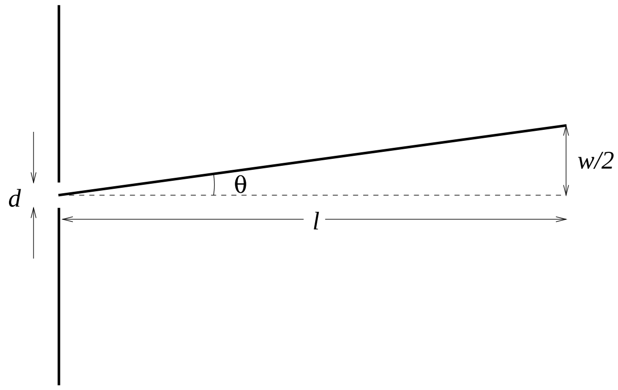
$$T \approx \frac{2 \times 1.32 \times 10^{-20}}{3 \times 1.38 \times 10^{-23}} \approx 640\,\mathrm{K}\,.$$

4. The De Broglie wavelength of a 100 eV electron is given by:

$$100\,\mathrm{eV} = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}\,,$$

and hence

$$\lambda = \frac{6.63 \times 10^{-34}}{\sqrt{2 \times 9.11 \times 10^{-31} \times 100 \times 1.60 \times 10^{-19}}} \approx 1.23 \times 10^{-10}\,\mathrm{m}\,.$$



From Q8 of Problem Sheet 1, the first zero in the diffraction pattern from a slit of width d occurs where $\sin\theta = \lambda/d$. Hence,

$$\theta = \sin^{-1}(\lambda/d) \approx \sin^{-1}(1.23 \times 10^{-10}/10^{-6}) \approx 1.23 \times 10^{-4}\,\text{radians}\,.$$

The width w of the central diffraction peak is $2l\tan\theta$, where $l = 1\,\mathrm{m}$ is the distance from the slit to the screen. Hence,

$$w = 2 \times 1 \times \tan\theta \approx 2\theta \approx 2.46 \times 10^{-4}\,\mathrm{m}\,.$$

5. A particle of mass m and momentum p has kinetic energy $p^2/2m$. If the kinetic energy is equal to $3k_BT/2$:

$$\frac{p^2}{2m} = \frac{3k_BT}{2}\,,$$

then

$$p = \sqrt{3mk_BT}\,.$$

Combining this result with De Broglie's equation, $p = h/\lambda$, gives:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{3mk_BT}}\,,$$

as required.

Avogadro's number of He atoms occupy a volume of $27.6 \times 10^{-6}\,\mathrm{m}^3$. Hence, the volume per atom d^3 is

$$\frac{27.6 \times 10^{-6}}{6.02 \times 10^{23}} \approx 4.58 \times 10^{-29}\,\mathrm{m}^3 \ .$$

Taking the cube root, we obtain $d \approx 3.58 \times 10^{-10}\,\mathrm{m}$.

To find the temperature T at which $\lambda = d$, we have to solve the equation

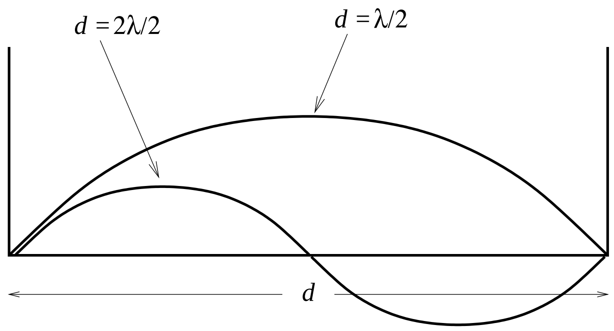
$$\frac{h}{\sqrt{3mk_BT}} = d \ .$$

Hence

$$\begin{aligned} T &= \frac{1}{3mk_B}\left(\frac{h}{d}\right)^2 \approx \frac{1}{3 \times 4 \times 1.66 \times 10^{-27} \times 1.38 \times 10^{-23}}\left(\frac{6.63 \times 10^{-34}}{3.58 \times 10^{-10}}\right)^2 \\ &\approx 12.5\,\mathrm{K} \ . \end{aligned}$$

When the temperature is comparable to or smaller than this value, the De Broglie wavelength of the He atoms will be the same as or greater than the interparticle spacing, and the wave-like properties of the atoms will be important.

6. The figure below shows that in order for the De Broglie wave of wavelength λ to "fit in" to the box, the box side d must be an integer multiple of $\lambda/2$: $d = n\lambda/2$, where $n = 1, 2, \ldots$



The maximum possible De Broglie wavelength is therefore $2d$. The smallest possible momentum is

$$p_{\mathrm{min}} = \frac{h}{\lambda_{\mathrm{max}}} = \frac{h}{2d} \approx \frac{6.63 \times 10^{-34}}{2 \times 3.58 \times 10^{-10}} \approx 9.26 \times 10^{-25}\,\mathrm{kg\,ms^{-1}} \ .$$

The smallest possible KE is

$$\mathrm{KE}_{\mathrm{min}} = \frac{p_{\mathrm{min}}^2}{2m} \approx \frac{(9.26 \times 10^{-25})^2}{2 \times 4 \times 1.66 \times 10^{-27}} \approx 6.46 \times 10^{-23}\,\mathrm{J} \ .$$

The thermal KE of $3k_BT/2$ would equal $\mathrm{KE}_{\mathrm{min}}$ when

$$T = \frac{2\,\mathrm{KE}_{\mathrm{min}}}{3k_B} \approx \frac{2 \times 6.46 \times 10^{-23}}{3 \times 1.38 \times 10^{-23}} \approx 3.12\,\mathrm{K} \ .$$

The Bohr Atom

7. The Rydberg formula is

$$\frac{1}{\lambda} = \mathrm{R_H}\left(\frac{1}{n_\mathrm{f}^2} - \frac{1}{n_\mathrm{i}^2}\right) ,$$

where $\mathrm{R_H} = 1.097 \times 10^7\,\mathrm{m}^{-1}$. In this case, $n_\mathrm{f} = 2$ and $\lambda = c/\nu = (3.00 \times 10^8)/(7.316 \times 10^{14}) = 4.10 \times 10^{-7}\,\mathrm{m}$.

Rearranging the Rydberg formula gives:

$$n_\mathrm{i}^2 = \left(\frac{1}{n_\mathrm{f}^2} - \frac{1}{\mathrm{R_H}\lambda}\right)^{-1} \approx \left(\frac{1}{4} - \frac{1}{1.097 \times 10^7 \times 4.10 \times 10^{-7}}\right)^{-1} \approx 36.1 .$$

Hence, the initial energy level was the $n = 6$ level.

8. The Bohr orbit of the electron must still contain a whole number of De Broglie wavelengths. The angular momentum $L = mvr$ must therefore be quantised just as in a hydrogen atom:

$$L = n\hbar , \qquad n = 1, 2, 3, \ldots .$$

However, since the Coulomb attraction between the orbiting electron and the nucleus is larger by a factor Z than in a hydrogen atom, the equation linking centripetal force and centripetal acceleration becomes:

$$\frac{Ze^2}{4\pi\epsilon_0 r^2} = \frac{mv^2}{r} = \frac{(mvr)^2}{mr^3} .$$

Replacing mvr by $n\hbar$ and rearranging gives the following formula,

$$r_n = \frac{4\pi\epsilon_0 (n\hbar)^2}{Zme^2} ,$$

for the radius of the n^th Bohr orbit.

The total energy of this orbit is

$$\begin{aligned}
E_n &= \mathrm{KE}_n + \mathrm{PE}_n \\
&= \frac{1}{2}mv_n^2 - \frac{Ze^2}{4\pi\epsilon_0 r_n} \\
&= \frac{1}{2}\frac{Ze^2}{4\pi\epsilon_0 r_n} - \frac{Ze^2}{4\pi\epsilon_0 r_n} && \left(\text{since } \frac{mv_n^2}{r_n} = \frac{Ze^2}{4\pi\epsilon_0 r_n^2}\right) \\
&= -\frac{Z^2 me^4}{2(4\pi\epsilon_0\hbar)^2 n^2} && (\text{substituting for } r_n) \\
&\approx -\frac{Z^2 \times 13.6\,\mathrm{eV}}{n^2} .
\end{aligned}$$

9. (i) The shortest wavelength (highest energy) photon that a hydrogen atom can emit without ending up in the ground-state ($n_\mathrm{f} = 1$) energy level is produced when the

atom decays from an initial state with very large $n_{\rm i}$ to the $n_{\rm f} = 2$ final state. The wavelength of the photon emitted in this transition satisfies the equation

$$\frac{1}{\lambda} = \mathrm{R_H}\left(\frac{1}{4} - \frac{1}{n_{\rm i}^2}\right) \approx \frac{\mathrm{R_H}}{4} ,$$

and hence

$$\lambda \approx \frac{4}{\mathrm{R_H}} \approx 3.65 \times 10^{-7}\,\mathrm{m} .$$

(ii) For the H atom, the normal Rydberg formula applies:

$$\frac{1}{\lambda} = \mathrm{R_H}\left(\frac{1}{n_{\rm f}^2} - \frac{1}{n_{\rm i}^2}\right) ,$$

with $\lambda = 121.5\,\mathrm{nm}$. Hence

$$\frac{1}{n_{\rm f}^2} - \frac{1}{n_{\rm i}^2} = \frac{1}{\lambda \mathrm{R_H}} \approx \frac{1}{121.5 \times 10^{-9} \times 1.097 \times 10^7} \approx 0.75 .$$

The only solution of this equation with $n_{\rm i}$ and $n_{\rm f}$ integers is $n_{\rm i} = 2$ and $n_{\rm f} = 1$. In other words, the transition is from the first excited state to the ground state.

For the He^+ ion, the energies of the states are $Z^2 = 2^2 = 4$ times what they were in the H atom (see Q8). Hence, the Rydberg formula becomes

$$\frac{1}{\lambda} = 4\mathrm{R_H}\left(\frac{1}{n_{\rm f}^2} - \frac{1}{n_{\rm i}^2}\right) ,$$

and the equation for $n_{\rm f}$ and $n_{\rm i}$ is

$$\frac{1}{n_{\rm f}^2} - \frac{1}{n_{\rm i}^2} \approx \frac{0.75}{4} = \frac{3}{16} .$$

This equation can be solved by doubling the values of $n_{\rm i}$ and $n_{\rm f}$ obtained for the H atom. In other words, the He^+ transition is from the $n_{\rm i} = 4$ level to the $n_{\rm f} = 2$ level.

10. According to the Bohr model, the binding energy and radius of an H atom in its ground state are

$$E_{\rm binding} = \frac{me^4}{2(4\pi\epsilon_0\hbar)^2} \approx 13.6\,\mathrm{eV} ,$$
$$r = \frac{4\pi\epsilon_0\hbar^2}{me^2} \approx 0.53\,\text{Å} .$$

In the case of a positronium atom, the electron mass m has to be replaced by the reduced mass $m_{\rm reduced} = (1/m + 1/m)^{-1} = 0.5m$. Hence

$$E_{\rm binding} = \frac{0.5me^4}{2(4\pi\epsilon_0\hbar)^2} \approx 6.8\,\mathrm{eV} ,$$
$$r = \frac{4\pi\epsilon_0\hbar^2}{0.5me^2} \approx 1.06\,\text{Å} .$$

11. (i) The potential in this example is peculiar, but a Bohr orbit of radius r still has length $2\pi r$, and the De Broglie wavelength of the particle must still "fit in" to this length:

$$2\pi r = n\lambda \qquad (n \text{ any integer} > 0)\ .$$

Since $p = mv = h/\lambda$, this condition translates to

$$mvr = mv\frac{n\lambda}{2\pi} = p\frac{n\lambda}{2\pi} = \frac{nh}{2\pi} = n\hbar\ .$$

In other words, the angular momentum must still be a multiple of $\hbar$.

The next step in deriving the Bohr model of the hydrogen atom is to write down Newton's second law: force = mass $\times$ centripetal acceleration. In this case, the force is

$$F = -\frac{dV}{dr} = -C\ ,$$

where the $-$ve sign shows that the force acts towards the origin (in the $-r$ direction). Hence, Newton's second law reads:

$$C = \frac{mv^2}{r} = \frac{(mvr)^2}{mr^3}\ .$$

Using the angular momentum quantisation conditions, $mvr = n\hbar$, then gives

$$C = \frac{\hbar^2 n^2}{mr_n^3}\ ,$$

or

$$r_n = \left(\frac{\hbar^2 n^2}{mC}\right)^{1/3}\ ,$$

as required.

(ii) The energy E_n of the n^{th} orbit is:

$$\begin{aligned} E_n &= \mathrm{KE} + \mathrm{PE} = \frac{1}{2}mv_n^2 + Cr_n \\ &= \frac{(mv_n r_n)^2}{2mr_n^2} + Cr_n = \frac{\hbar^2 n^2}{2mr_n^2} + Cr_n \\ &= \frac{\hbar^2 n^2}{2m\left(\frac{\hbar^2 n^2}{mC}\right)^{2/3}} + C\left(\frac{\hbar^2 n^2}{mC}\right)^{1/3} \\ &= \frac{C}{2}\left(\frac{\hbar^2 n^2}{mC}\right)^{1/3} + C\left(\frac{\hbar^2 n^2}{mC}\right)^{1/3} \\ &= \frac{3}{2}\left(\frac{C^2\hbar^2 n^2}{m}\right)^{1/3}\ , \end{aligned}$$

as required.