Thursday 10th March 2005

Quantum Physics Problem Sheet 2

Note that these questions are ordered by topic, not by difficulty. Do not get disheartened if you find some of the questions near the beginning of the sheet hard.

Compton Scattering

1. X-rays of energy 20 keV are Compton scattered by a thin metal foil through an angle of 60^o. Find (i) the wavelength of the scattered photons, and (ii) the energy (in eV) lost by each photon. Would the kinetic energy of the recoiling electrons be sufficient to allow some of them to escape from the metal?

   What value of the scattering angle θ would give the largest change in wavelength? What is the maximum possible wavelength of the scattered photon?

2. (Q40-63 from Young and Freedman) Nuclear fusion reactions at the centre of the sun produce gamma-ray photons with energies of order 1 MeV. By contrast, what we see emanating from the sun's surface are visible photons with wavelengths of order 500 nm. Models of the solar interior explain this wavelength difference by suggesting that every photon is Compton scattered about 10^{26} times during its journey from the centre of the sun to the surface.

   (i) Estimate the average increase in wavelength of a solar photon per Compton-scattering event.

   (ii) Find the angle in degrees through which the photon is scattered in the scattering event described in part (i). (Hint: a useful approximation is $\cos\phi \approx 1 - \phi^2/2$, which is valid when ϕ is $\ll 1$ radian.)

   (iii) It is estimated that a photon takes about 10^6 years to travel from the core to the surface of the sun. Find the average distance that light can travel within the interior of the sun without being scattered.

   As you can see, the sun is *very* opaque, and the radiation has plenty of opportunity to reach equilibrium with the matter before emerging. This explains why the sun emits black-body radiation.

De Broglie Waves

3. Neutron diffraction is often used to study the atomic positions and atomic-scale magnetic fields in solids. Suggest a reasonable value for the De Broglie wavelength of the neutrons used in such experiments. Find the kinetic energy (in eV) of neutrons of this wavelength. Use the equation $E = 3k_BT/2$ to translate the kinetic energy into an equivalent temperature.

4. Electrons of energy 100 eV pass through a narrow slit of width 1 μm. What is the distance between the zeros of intensity on either side of the central peak of the electron diffraction pattern 1 m away from the slit? (Hint: you worked out the form of the diffraction pattern in Q8 of Problem Sheet 1.)

5. Show that the De Broglie wavelength of a particle of mass m and kinetic energy $3k_BT/2$ is

$$\lambda = \frac{h}{\sqrt{3mk_BT}} \, .$$

   The molar volume of liquid ^{4}He is 27.6 cm^3. Assuming that each atom occupies a cube of side d, calculate the temperature at which $\lambda = d$. Discuss the temperature range over which the wave-like properties of the atoms in liquid ^{4}He are likely to be important.

   (Incidentally, liquid ^{4}He becomes a superfluid, able to flow without friction, below 2.17 K. The above calculation suggests that superfluidity is almost certainly a QM phenomenon.)

6. When liquid ^{4}He freezes, every atom is confined to a "box" (its lattice site in the crystal). Since liquids and solids normally have similar densities, the box size is similar to the value of d calculated in Q5. Assuming that the De Broglie wave of wavelength λ associated with the ^{4}He atom has to equal zero at the box walls, show (perhaps by drawing a diagram) that $d = n\lambda/2$, where n is any integer > 0. Hence calculate the smallest possible momentum and kinetic energy of the confined atom. At what temperature T would the thermal kinetic energy $3k_BT/2$ equal the quantum mechanical KE of confinement?

   > Note: The idea behind this question is important and very general. Since the De Broglie wavelength of a confined particle has to "fit in" to the confining box, the particle must have a non-zero momentum and KE. The particle must therefore be moving — rattling backwards and forwards inside the box — even at zero temperature. This confinement-induced motion is called zero-point motion, and the corresponding kinetic energy is called zero-point energy. The orbits of electrons in atoms can be viewed as a type of zero-point motion.
   >
   > In most solids, the atoms are heavy enough and the chemical bonds strong enough that the zero-point motion of the atoms (as opposed to the electrons) is unimportant. In ^{4}He, however, where the atoms are comparatively light and the bonding is very weak, the zero-point motion alone is sufficient to melt the solid — there is no need to heat it up! This is why liquid ^{4}He remains liquid right down to $T = 0$ K. To solidify ^{4}He, it is necessary to apply pressure.

The Bohr Atom

7. Light of frequency 7.316×10^{14} Hz is emitted in a downward transition to the $n = 2$ energy level of a hydrogen atom. What was the initial energy level?

8. Generalise the derivation of the Bohr model given in lectures to obtain a formula for the energy levels of ions such as He$^+$ or Al^{12+}, which have atomic number $Z > 1$ ($Z = 2$ for He; $Z = 13$ for Al) but only one orbiting electron.

9. (i) What is the *shortest* wavelength photon that could be emitted by a hydrogen atom that did not end up in the ground state $n = 1$ level.

   (ii) A spectral line at 121.5 nm is observed in both the H atom emission spectrum and, at exactly the same wavelength, in the spectrum from the singly ionized He^+ ion. Estimate the initial and final (n_i and n_f) levels of this transition in the H and He^+ spectra.

10. An electron and a positron (same mass, opposite charge) can form a short-lived bound state called a positronium atom, in which the two particles orbit their centre of mass. Use the Bohr model to estimate the binding energy and the distance between the electron and the positron in the positronium ground state. (Hint: consider a single particle of mass $m_\mathrm{reduced} = (1/m + 1/m)^{-1}$ orbiting a fixed origin.)

11. A particle of mass m is bound in a spherical potential of the form $V(r) = Cr$. By adapting the derivation of the Bohr model to this case, show that:

    (i) the quantised orbital radii are

$$r_n = \left(\frac{\hbar^2 n^2}{mC}\right)^{1/3} \qquad (n \text{ any integer} \geq 1) \; ;$$

    (ii) the quantised energy levels are

$$E_n = \frac{3}{2}\left(\frac{C^2\hbar^2 n^2}{m}\right)^{1/3} \qquad (n \text{ any integer} \geq 1) \; .$$

Physical Constants

$$\begin{aligned}
m_\mathrm{electron} &\approx 9.11 \times 10^{-31}\,\mathrm{J} \approx 511\,\mathrm{keV}/c^2 \\
m_\mathrm{neutron} &\approx 1.67 \times 10^{-27}\,\mathrm{kg} \\
\text{atomic mass unit} &\approx 1.66 \times 10^{-27}\,\mathrm{kg} \\
h &\approx 6.63 \times 10^{-34}\,\mathrm{Js} \\
\hbar &\approx 1.05 \times 10^{-34}\,\mathrm{Js} \\
c &\approx 3.00 \times 10^{8}\,\mathrm{ms}^{-1} \\
e &\approx 1.60 \times 10^{-19}\,\mathrm{C} \\
k_B &\approx 1.38 \times 10^{-23}\,\mathrm{JK}^{-1} \\
\mathrm{R_H} &\approx 1.097 \times 10^{7}\,\mathrm{m}^{-1}
\end{aligned}$$

Numerical Answers

1. (i) 6.34×10^{-11} m; (ii) 378 eV (with large and difficult to estimate numerical uncertainty). Angle for largest change in wavelength is $\theta = 180^o$; maximum possible scattered wavelength is 6.71×10^{-11} m.
2. (i) 5×10^{-33} m; (ii) 3.68×10^{-9} degrees; (iii) 9.46×10^{-5} m.
3. Any wavelength within an order of magnitude of the interatomic spacing OK. Choosing $\lambda = 10^{-10}$ m gives KE of 0.083 eV and temperature of 640 K.
4. 2.46×10^{-4} m.
5. $T = 12.5$ K.
6. $p_{\rm min} = 9.26 \times 10^{-25}\,{\rm kg\,m\,s^{-1}}$; ${\rm KE_{min}} = 6.46 \times 10^{-23}$ J; $T = 3.12$ K.
7. Initial energy level $n_{\rm i} = 6$.
9. (i) 3.65×10^{-7} m; (ii) For H, $n_{\rm i} = 2$ and $n_{\rm f} = 1$; for He, $n_{\rm i} = 4$ and $n_{\rm f} = 2$.
10. $E_{\rm binding} \approx 6.8$ eV; $r \approx 1.06$ Å.