Monday 28th February 2005

Quantum Physics Problem Sheet 1

Note that these questions are ordered by topic, not by difficulty. Do not get disheartened if you find some of the questions near the beginning of the sheet hard.

Units and Magnitudes

1. Metallic aluminium has atomic weight 27 and density $2700\,\mathrm{kg\,m^{-3}}$. Estimate the radius of an aluminium atom in Ångstroms.
2. Electrons (mass $m = 0.51\,\mathrm{MeV/c^2}$) in a television tube are accelerated through a potential difference of 25 kV. Using relativistic formulae and units (MeV/c^2 for energies, MeV/c for momenta, and so on), calculate the final speed of the electrons as a fraction of the speed of light. Are relativistic effects important?
3. The ionisation energy of a hydrogen atom is 13.6 eV. By equating the thermal energy scale, $k_B T$, to the ionisation energy, estimate the temperature of the universe when the first neutral H atoms formed.

Travelling Waves

4. Without referring to your lecture notes, show that the wave

$$\psi(x,t) = a\cos(-kx - \omega t + \phi)$$

has wavelength $\lambda = 2\pi/k$ and frequency $\nu = \omega/2\pi$. Show that the wave crests travel with speed ω/k in the $-x$ direction.

5. The dispersion relation of large deep ocean waves is $\omega = \sqrt{gk}$. What is the phase velocity (the velocity of the wave crests) when the wavelength is 10 m?

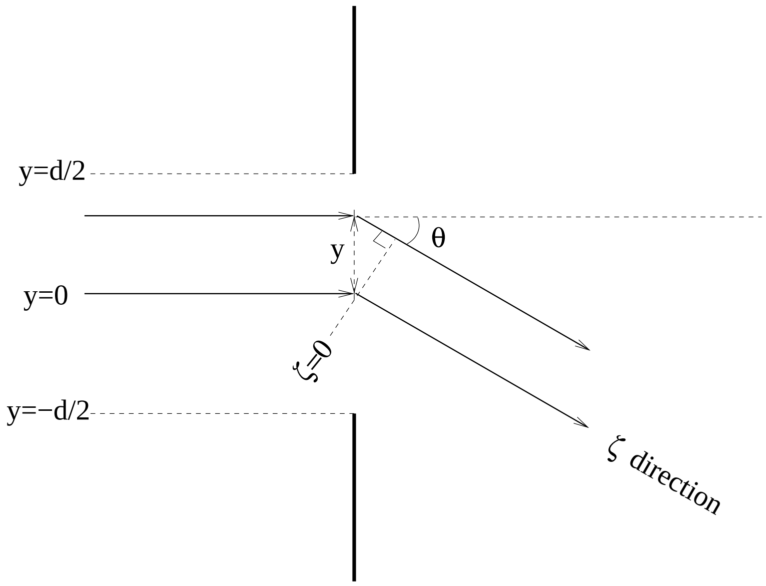
Complex Representation of Waves, Interference and Diffraction

6. Use complex numbers to show that $\cos\theta + \sin\theta = \sqrt{2}\cos(\theta - \pi/4)$. [Hint: $\sin\theta = \mathrm{Re}(-ie^{i\theta})$.]
7. Consider the function

$$\psi(x,t) = a\cos(kx - \omega t) + a\cos(-kx - \omega t)$$

obtained by superposing right- and left-going travelling waves of equal amplitude. Using (i) real numbers only, and (ii) complex numbers, show that $\psi(x,t)$ is a standing wave and calculate its intensity $I(x)$. Show that the position average of $I(x)$ is $2a^2$. Why is this result expected?

8. This question takes you through the theory of diffraction — the way that waves spread out as they emerge from a narrow opening. Although quite difficult, it provides good practice with the complex representation of waves and will prove important when we discuss the uncertainty principle. It is worth the effort.



Parallel monochromatic plane waves of wavelength $\lambda = 2\pi/k$ and frequency $\nu = \omega/2\pi$ pass through a narrow slit stretching from $y = -d/2$ to $y = d/2$. To work out the wave emerging at angle θ, it is helpful to imagine dividing the slit into huge numbers of tiny segments, each of height Δy. The amplitude of the wave emerging from each segment is proportional to the height of that segment, and so the wave emerging from the segment at $y = 0$ may be written as

$$Ae^{i(k\zeta-\omega t)}\Delta y \ ,$$

where ζ measures the distance from the centre of the slit in the θ direction and the complex constant A encodes the overall amplitude and phase of the wave.

(i) Write down the wave emerging from the segment Δy at height y.

(ii) Show that the total wave emerging in the ζ direction is

$$\psi(\zeta,t) = Ae^{i(k\zeta-\omega t)} \int_{-d/2}^{d/2} e^{iky\sin\theta} dy \ .$$

(iii) Evaluate the integral and hence show that the intensity emerging at angle θ is

$$I(\theta) = \frac{|A|^2 d^2 \sin^2\left(\frac{kd\sin\theta}{2}\right)}{\left(\frac{kd\sin\theta}{2}\right)^2} \ .$$

(iv) Sketch I as a function of $\frac{1}{2}kd\sin\theta$. What is the path-length difference between the top and the bottom of the slit when θ corresponds to the first zero of the diffraction pattern?

Photons

9. The energy flux of starlight reaching us from a sixth-magnitude star (approximately the faintest that can be seen by the naked eye) is 1.4×10^{-10} Wm^{-2}. If you are looking at such a star, how many photons enter your eye per second? On average, how many photons are inside your eye at any one time? Do these estimates provide any evidence that the human eye can detect single photons? (Assume that the diameter of your dark-adapted pupil is 0.7 cm, the length of your eye is 4 cm, and the wavelength of the light is 500 nm.)

10. Electrons in an X-ray tube are accelerated by a potential difference of 30 kV. What is the minimum wavelength of the resulting X-rays?

11. In a photoelectric effect experiment, a current was observed when the metallic cathode was illuminated with light of wavelength 310 nm, but none was observed with longer wavelength light.

    (i) Estimate the work function of the cathode in eV.

    (ii) What is the energy (in eV) of a photon of wavelength 200 nm?

    (iii) Find the stopping potential (in Volts) at 200 nm. Hence find the maximum kinetic energy (in eV) of the electrons emitted at this wavelength.

12. In a photoelectric effect experiment, the cathode was illuminated with EM radiation of three different frequencies and the stopping potential V_0 was measured for each.

Frequency ν (10^{15} Hz)	Stopping Potential V_0 (V)
0.75	1.0
1.00	2.0
1.25	3.0

    (i) Make a sketch of stopping potential against frequency and hence estimate the work function of the metal in eV.

    (ii) From the slope of your line, estimate the value of Planck's constant h. How close is this to the accepted value?

Physical Constants

$$\begin{aligned}
m_{\text{electron}} &\approx 9.11 \times 10^{-31}\,\text{J} \approx 511\,\text{keV}/c^2 \\
\text{atomic mass unit} &\approx 1.66 \times 10^{-27}\,\text{kg} \\
h &\approx 6.63 \times 10^{-34}\,\text{Js} \\
\hbar &\approx 1.05 \times 10^{-34}\,\text{Js} \\
c &\approx 3.00 \times 10^{8}\,\text{ms}^{-1} \\
e &\approx 1.60 \times 10^{-19}\,\text{C} \\
g &\approx 9.8\,\text{ms}^{-1} \\
N_{\text{A}} &\approx 6.02 \times 10^{23} \\
R &\approx 8.314\,\text{JK}^{-1} \\
k_B &\approx 1.38 \times 10^{-23}\,\text{JK}^{-1}
\end{aligned}$$

Numerical Answers

1. Any radius between 1.25 Å and 1.6 Å acceptable.
2. $v/c \approx 0.30$.
3. $160,000\,\text{K}$.
5. $3.96\,\text{ms}^{-1}$.
9. Number of photons entering eye per second $\approx 13,500$; expected number inside eye at any one time $\approx 1.8 \times 10^{-6}$.
10. $4.14 \times 10^{-11}\,\text{m}$.
11. (i) 4.00 eV; (ii) 6.22 eV; (iii) Stopping potential 2.22 V; maximum KE 2.22 eV.