Tuesday 3rd May 2005

Answer to Quantum Physics Classwork 4

The Bohr Model

1. (a) In the Bohr model, the angular momentum L of the orbiting electron is a multiple of $\hbar$:

$$L = m_e v r = n\hbar \qquad (n \text{ any integer} > 0) .$$

The orbiting electron satisfies Newton's second law: force = mass × (centripetal acceleration). The force is the Coulomb attraction between the electron (charge $-e$) and the nucleus (charge Ze). Hence:

$$\frac{Ze^2}{4\pi\epsilon_0 r^2} = \frac{m_e v^2}{r} = \frac{(m_e v r)^2}{m_e r^3} = \frac{(n\hbar)^2}{m_e r^3} .$$

This gives:

$$r_n = \frac{4\pi\epsilon_0 \hbar^2 n^2}{Ze^2 m_e} ,$$

as required.

(b) The KE of the n^{th} orbit is:

$$\frac{1}{2} m_e v_n^2 = \frac{(m_e v_n r_n)^2}{2 m_e r_n^2} = \frac{n^2\hbar^2}{2m_e\left(\frac{4\pi\epsilon_0\hbar^2 n^2}{Ze^2 m_e}\right)^2} = \frac{Z^2 m_e e^4}{2(4\pi\epsilon_0 \hbar n)^2} .$$

The PE is:

$$-\frac{Ze^2}{4\pi\epsilon_0 r_n} = -\frac{Ze^2}{4\pi\epsilon_0\left(\frac{4\pi\epsilon_0\hbar^2 n^2}{Ze^2 m_e}\right)} = -\frac{Z^2 m_e e^4}{(4\pi\epsilon_0\hbar n)^2} .$$

Hence the total energy is:

$$E_n = KE_n + PE_n = -\frac{Z^2 m_e e^4}{2(4\pi\epsilon_0\hbar n)^2} .$$

(c) We seek the value of Z such that $KE_1 = 0.1\, m_e c^2$:

$$\frac{Z^2 m_e e^4}{2(4\pi\epsilon_0\hbar)^2} = 0.1 \times 0.511 \times 10^6\,\text{eV}$$

$$Z^2 \times 13.6\,\text{eV} = 5.11 \times 10^4\,\text{eV} .$$

This gives $Z = 61$, rounded to the nearest integer. The KE increases with Z and decreases with n (core electrons orbit fastest). Hence, relativistic effects are important for the core states of atoms with atomic number $Z \gtrsim 60$.

(d) The energy of the photon is:

$$E = \frac{hc}{\lambda} = 8.65 \times 10^{-18}\,\mathrm{J} = \frac{8.65 \times 10^{-18}}{1.60 \times 10^{-19}}\,\mathrm{eV} = 54.05\,\mathrm{eV} \ .$$

This is approximately $4 \times 13.6\,\mathrm{eV}$ and so the nucleus has $Z = 2$. It is a He nucleus.

2. (a) Since deuterons have charge e, the Coulomb energy barrier is:

$$\frac{e^2}{4\pi\epsilon_0 r_{\mathrm{nuc}}} = \frac{e^2}{4\pi\epsilon_0(5 \times 10^{-15})} \approx 4.6 \times 10^{-14}\,\mathrm{J} \approx 290\,\mathrm{keV} \ .$$

The nuclear KE of $3k_BT/2$ is equal to the barrier height when:

$$T \approx \frac{2 \times 4.6 \times 10^{-14}}{3k_B} \approx 2.2 \times 10^{9}\,\mathrm{K} \ .$$

This is *very* hot. Fortunately, since some of the particles in a Maxwell-Boltzmann distribution are travelling much faster than the average speed, some fusion can take place at much lower temperatures. In fact, the ideal operating temperature of a tokamak corresponds to an average particle energy of only (!) 10–20 keV.

(b) According to the result of question 1(a), the atomic radius is inversely proportional to the mass of the orbiting particle.

(i) If the orbiting particle is an electron, the orbital radius is 1 Bohr radius. Hence, the fusion rate is:

$$e^{40-210} = 10^{-170\log_{10} e} \approx 10^{-74}\,\mathrm{s}^{-1} \ .$$

(ii) If the orbiting particle is a muon of mass $207m_e$, the orbital radius is $a_0/207$. Hence, the fusion rate is:

$$e^{40-\frac{210}{\sqrt{207}}} = 10^{\left(40-\frac{210}{\sqrt{207}}\right)\log_{10} e} \approx 10^{11}\,\mathrm{s}^{-1} \ .$$

Replacing the electron by a muon increases the tunnelling rate by 85 orders of magnitude!

(c) To take account of the finite mass of the nucleus, all that is necessary is to replace the muon mass m_μ by the reduced mass

$$\begin{aligned} m_{\rm red} &= \left(\frac{1}{m_\mu} + \frac{1}{m_d}\right)^{-1} = m_\mu \left(1 + \frac{m_\mu}{m_d}\right)^{-1} \\ &= m_\mu \left(1 + \frac{207}{3760}\right)^{-1} \approx 0.948 m_\mu \, . \end{aligned}$$

Since $r_n \propto 1/m$, the distance between the deuteron and the muon increases by the factor $1/0.948 \approx 1.055$. Since $E_n \propto m$, the magnitude of the energy decreases by the factor 0.948 (in other words, the muon is slightly less strongly bound).