Tuesday 3rd May 2005

Quantum Physics Classwork 4

The Bohr Model

1. In lectures, the Bohr model was applied to the H atom. This classwork applies it to one-electron ions such as He^+, Li^{2+} and Mn^{24+}.

   (a) Using the Bohr model, show that the orbital radii of the electron in a one-electron ion of atomic number Z are

   $$r_n = \frac{4\pi\epsilon_0\hbar^2 n^2}{Ze^2 m_e} \ ,$$

   where n is any positive integer.

   (b) Derive expressions for the kinetic energy and potential energy of an electron in the n^{th} orbit. Hence show that the total energy is

   $$E_n = -\frac{Z^2 m_e e^4}{2(4\pi\epsilon_0\hbar n)^2} \ .$$

   (c) Find the value of Z (rounded to the nearest integer) at which the kinetic energy of an electron in the $n = 1$ orbit is equal to $0.1\, m_e c^2$? Where in the periodic table are relativistic effects important? Which quantum states are most affected?

   (d) A bare nucleus captures an electron and emits a photon. Given that the minimum observed photon wavelength is approximately 2.3×10^{-8} m, identify the nucleus.

2. When light nuclei such as deuterons (1 proton + 1 neutron) and tritons (1 proton + 2 neutrons) fuse to form a heavier nucleus, large amounts of energy are released. For fusion to occur, however, it is necessary to bring the two nuclei to within approximately 5×10^{-15} m of each other. This is difficult because of the Coulomb repulsion. One method, used in hydrogen bombs and tokamaks (a type of fusion reactor), is to heat the nuclei until they have enough kinetic energy to overcome the Coulomb barrier.

   (a) Roughly how hot would a plasma of electrons and deuterons have to be for the average nuclear kinetic energy to exceed the Coulomb barrier? Would a tokamak have to reach this enormously high temperature in order to work?

An alternative idea, proposed by Zel'dovich in the fifties, is to work at low temperature and rely on quantum mechanical tunnelling (which will be explained later in the course) to bring the nuclei together. For an ordinary D_2 molecule, the fusion rate due to tunnelling is far too small to be useful. Zel'dovich's idea was to replace the electrons by muons, which have the same charge as an electron but 207 times the mass. This makes the atoms shrink, bringing the deuterons closer together and enhancing the fusion rate dramatically.

(b) A (very) rough expression for the fusion rate of a D_2 molecule is $\exp(40 - 210\sqrt{a})\,\mathrm{s}^{-1}$, where a is the atomic radius measured in Bohr radii. Estimate the reaction rates for D_2 molecules containing (i) electrons and (ii) muons.

Although muon-catalysed fusion does work, the bad news is that muons cost energy to produce and have a very short half-life. The current consensus is that the total fusion energy liberated per muon is smaller than the energy required to produce the muon in the first place.

(c) Since the mass of a muon ($207m_e$) is a significant fraction of the mass of a deuteron ($3670m_e$), it is not a good approximation to assume that the deuteron remains stationary as the muon orbits around it. Instead, both particles orbit their mutual centre of mass. How does this affect the orbital radii and the atomic energy levels? (Hint: the reduced mass m_{red} of two masses m_1 and m_2 is given by $1/m_{\mathrm{red}} = 1/m_1 + 1/m_2$.)

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\begin{aligned}
\frac{4\pi\epsilon_0\hbar^2}{e^2 m_e} &\approx 0.529\times10^{-10}\,\mathrm{m}\ \ \text{(Bohr radius)}\\
\frac{m_e e^4}{2(4\pi\epsilon_0\hbar)^2} &\approx 13.6\,\mathrm{eV}\ \ \text{(Ionisation energy of H atom)}\\
m_e &\approx 0.511\,\mathrm{MeV}/c^2\\
h &\approx 6.63\times10^{-34}\,\mathrm{Js}\\
c &\approx 3.00\times10^{8}\,\mathrm{ms}^{-1}\\
e &\approx 1.60\times10^{-19}\,\mathrm{C}\\
\epsilon_0 &\approx 8.85\times10^{-12}\,\mathrm{Fm}^{-1}\\
k_B &\approx 1.38\times10^{-23}\,\mathrm{JK}^{-1}
\end{aligned}
$$