Monday 21st February 2005

Answer to Quantum Physics Classwork 1
Why Represent Wave with Complex Numbers?

Real Version

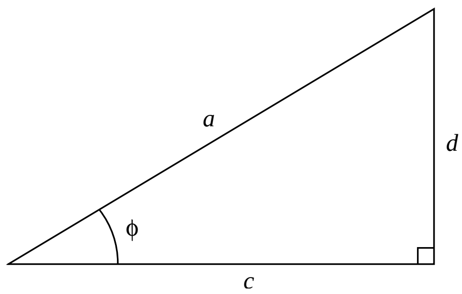
1.
$$\begin{aligned}
\psi(x,t) &= a_1\cos(kx-\omega t+\phi_1)+a_2\cos(kx-\omega t+\phi_2) \\
&= a_1\left[\cos(kx-\omega t)\cos\phi_1-\sin(kx-\omega t)\sin\phi_1\right] \\
&\qquad + a_2\left[\cos(kx-\omega t)\cos\phi_2-\sin(kx-\omega t)\sin\phi_2\right] \\
&= (a_1\cos\phi_1+a_2\cos\phi_2)\cos(kx-\omega t) \\
&\qquad - (a_1\sin\phi_1+a_2\sin\phi_2)\sin(kx-\omega t) \\
&= c\cos(kx-\omega t)-d\sin(kx-\omega t)\ ,
\end{aligned}$$

   where

$$c=a_1\cos\phi_1+a_2\cos\phi_2 \qquad \text{and} \qquad d=a_1\sin\phi_1+a_2\sin\phi_2$$

   as required.

2. From the right-angle triangle



   we see that $c=a\cos\phi$ and $d=a\sin\phi$, where

$$a=\sqrt{c^2+d^2} \qquad \text{and} \qquad \phi=\tan^{-1}(d/c)\ .$$

   Hence

$$\begin{aligned}
\psi(x,t) &= c\cos(kx-\omega t)-d\sin(kx-\omega t) \\
&= a[\cos\phi\cos(kx-\omega t)-\sin\phi\sin(kx-\omega t)]\ ,
\end{aligned}$$

   as required.

3. Using the identity $\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \cos(\theta_1+\theta_2)$, the above expression for ψ becomes

$$\psi(x,t) = a\cos(kx - \omega t + \phi) \ .$$

From the triangle that defines a we have $a^2 = c^2 + d^2$. Combining this with the expressions for c and d in terms of a_1, a_2, ϕ_1 and ϕ_2 gives

$$\begin{aligned} a^2 &= (a_1\cos\phi_1 + a_2\cos\phi_2)^2 + (a_1\sin\phi_1 + a_2\sin\phi_2)^2 \\ &= a_1^2(\cos^2\phi_1 + \sin^2\phi_1) + a_2^2(\cos^2\phi_2 + \sin^2\phi_2) \\ &\quad + 2a_1a_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) \\ &= a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2) \ , \end{aligned}$$

where the last step used the result $\cos(\phi_1 - \phi_2) = \cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2$ given in the classwork. Hence

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2)} \ ,$$

as required.

The triangle also shows that $\tan\phi = d/c$. Combining this with the expressions for c and d in terms of a_1, a_2, ϕ_1 and ϕ_2 gives

$$\phi = \tan^{-1}\left(\frac{a_1\sin\phi_1 + a_2\sin\phi_2}{a_1\cos\phi_1 + a_2\cos\phi_2}\right) \ ,$$

as required.

Complex Version

4.

$$\begin{aligned} \tilde{\psi}(x,t) &= a_1e^{i(kx-\omega t+\phi_1)} + a_2e^{i(kx-\omega t+\phi_2)} \\ &= a_1e^{i\phi_1}e^{i(kx-\omega t)} + a_2e^{i\phi_2}e^{i(kx-\omega t)} \\ &= (a_1e^{i\phi_1} + a_2e^{i\phi_2})e^{i(kx-\omega t)} \\ &= Ae^{i(kx-\omega t)} \ , \end{aligned}$$

where $A = a_1e^{i\phi_1} + a_2e^{i\phi_2}$ as required.

5. Since $a = \sqrt{A^* A}$, we have

$$\begin{aligned} a^2 &= (a_1 e^{-i\phi_1} + a_2 e^{-i\phi_2})(a_1 e^{i\phi_1} + a_2 e^{i\phi_2}) \\ &= a_1^2 + a_2^2 + a_1 a_2 \left(e^{i(\phi_1 - \phi_2)} + e^{-i(\phi_1 - \phi_2)} \right) \\ &= a_1^2 + a_2^2 + 2 a_1 a_2 \cos(\phi_1 - \phi_2) \ , \end{aligned}$$

where the last step used the result $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ given in the classwork.

Since $\phi = \tan^{-1}(\mathrm{Re}(A)/\mathrm{Im}(A))$, we have

$$\begin{aligned} \phi &= \tan^{-1}\left(\frac{\mathrm{Re}(a_1 e^{i\phi_1} + a_2 e^{i\phi_2})}{\mathrm{Im}(a_1 e^{i\phi_1} + a_2 e^{i\phi_2})} \right) \\ &= \tan^{-1}\left(\frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \sin\phi_2} \right) \ , \end{aligned}$$

where the last step used the result $e^{i\theta} = \cos\theta + i \sin\theta$ given in the classwork.

6.
$$\begin{aligned} \psi(x,t) &= \mathrm{Re}(A e^{i(kx - \omega t)}) \\ &= \mathrm{Re}\left(a e^{i\phi} e^{i(kx-\omega t)} \right) \\ &= \mathrm{Re}\left(a e^{i(kx - \omega t + \phi)} \right) \\ &= a \cos(kx - \omega t + \phi) \ , \end{aligned}$$

where the last step used the result $e^{i\theta} = \cos\theta + i \sin\theta$ given in the classwork.