Monday 21st February 2005

Quantum Physics Classwork 1

Why Represent Waves with Complex Numbers?

Review

As you already know, interference patterns are created when waves overlap. The total displacement is the sum of the displacements of the contributing waves, and the interference pattern is the intensity corresponding to the total displacement. When working out interference patterns, the first step is to add together the contributing waves to find the total displacement.

Suppose, for example, that you want to work out the sum $\psi(x,t)$ of two travelling waves:

$$\begin{aligned}\psi_1(x,t) &= a_1\cos(kx-\omega t+\phi_1)\ ,\\ \psi_2(x,t) &= a_2\cos(kx-\omega t+\phi_2)\ .\end{aligned}$$

The obvious approach is to use trigonometic identities to simplify the expression for $\psi_1(x,t)+\psi_2(x,t)$, but it is often easier to view each contributing wave as the real part of a complex wave,

$$\begin{aligned}\psi_1(x,t) &= \mathrm{Re}(\tilde{\psi}_1(x,t)) &= \mathrm{Re}(a_1 e^{i(kx-\omega t+\phi_1)})\ ,\\ \psi_2(x,t) &= \mathrm{Re}(\tilde{\psi}_2(x,t)) &= \mathrm{Re}(a_2 e^{i(kx-\omega t+\phi_2)})\ ,\end{aligned}$$

and to work out

$$\tilde{\psi}(x,t) = \tilde{\psi}_1(x,t) + \tilde{\psi}_2(x,t)$$

using complex arithmetic. Since

$$\mathrm{Re}(\tilde{\psi}) = \mathrm{Re}(\tilde{\psi}_1+\tilde{\psi}_2) = \mathrm{Re}(\tilde{\psi}_1)+\mathrm{Re}(\tilde{\psi}_2) = \psi_1+\psi_2\ ,$$

the real part of the complex answer $\tilde{\psi}(x,t)$ is the physical answer $\psi_1(x,t)+\psi_2(x,t)$.

In classical physics, the displacements are always real and the complex representation is simply a mathematical trick to make the algebra easier. In quantum theory, however, the wave function is *really* complex (there is an i in the Schrödinger equation) and the use of complex waves is inescapable.

This classwork revises the use of the complex representation in classical physics. By asking you to work out $\psi(x,t)$ using both the real representation and the complex representation, I hope to convince you that the complex representation is a good thing.

Real Version

1. Show that

$$\begin{aligned}\psi(x,t) &= a_1\cos(kx-\omega t+\phi_1)+a_2\cos(kx-\omega t+\phi_2)\\ &= c\cos(kx-\omega t)-d\sin(kx-\omega t)\ ,\end{aligned}$$

   where $c = a_1\cos\phi_1 + a_2\cos\phi_2$ and $d = a_1\sin\phi_1 + a_2\sin\phi_2$. (There is a table of trigonometric formulae at the end of the classwork.)

2. Draw a right-angled triangle with c and d as the lengths of the adjacent and opposite sides. Hence show that

$$\psi(x,t) = a\left[\cos\phi\cos(kx-\omega t)-\sin\phi\sin(kx-\omega t)\right]$$

   where

$$a = \sqrt{c^2+d^2} \qquad \text{and} \qquad \phi = \tan^{-1}(c/d)\ .$$

3. Show that $\psi(x,t) = a\cos(kx-\omega t+\phi)$, where

$$a = \sqrt{a_1^2+a_2^2+2a_1a_2\cos(\phi_1-\phi_2)}$$

   and

$$\phi = \tan^{-1}\left(\frac{a_1\sin\phi_1+a_2\sin\phi_2}{a_1\cos\phi_1+a_2\cos\phi_2}\right)\ .$$

Complex Version

4. Show that

$$\tilde{\psi}(x,t) = a_1e^{i(kx-\omega t+\phi_1)} + a_2e^{i(kx-\omega t+\phi_2)}$$

   may be written as $Ae^{i(kx-\omega t)}$, where $A = a_1e^{i\phi_1} + a_2e^{i\phi_2}$.

5. The complex number A may be written in polar form as $ae^{i\phi}$, where $a = |A|$ and $\phi = \tan^{-1}(\mathrm{Im}A/\mathrm{Re}A)$. Show that

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2)}$$

and

$$\phi = \tan^{-1}\left(\frac{a_1\sin\phi_1 + a_2\sin\phi_2}{a_1\cos\phi_1 + a_2\cos\phi_2}\right)$$

6. Show that

$$\psi(x,t) = \mathrm{Re}(\tilde{\psi}(x,t)) = \mathrm{Re}(Ae^{i(kx-\omega t)}) = a\cos(kx - \omega t + \phi)\ .$$

As you can see, even in this simple example, the complex method is easier than the real method; in complicated examples with many contributing waves, the complex method is *much* easier.

Some or all of the following trigonometric identities may be useful:

$$\begin{aligned}
\cos(\theta_1 \pm \theta_2) &= \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2 \\
\sin(\theta_1 \pm \theta_2) &= \sin\theta_1\cos\theta_2 \pm \sin\theta_2\cos\theta_1 \\
e^{\pm i\theta} &= \cos\theta \pm i\sin\theta \\
\cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
\sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}
\end{aligned}$$