Quantum Physics Answer Sheet 1

Units and Magnitudes

1. Number of atoms per unit volume is:

$$n = \frac{\text{Mass per unit volume}}{\text{Mass per atom}} = \frac{2700}{27 \times 1.66 \times 10^{-27}} = 6.02 \times 10^{28}\,\text{m}^{-3}\ .$$

Volume per atom is $1/n = 1.66 \times 10^{-29}\,\text{m}^3$. If we assume that each atom is a sphere of radius r, we have

$$\frac{4}{3}\pi r^3 = 1.66 \times 10^{-29}$$

and hence $r = 1.58 \times 10^{-10}\,\text{m} = 1.58\,\text{Å}$. This is an overestimate because it is impossible to fill space with spheres (there are always gaps between them). A different estimate may be obtained by assuming that the spheres are stacked in a simple cubic lattice. Each cube of side $2r$ then contains one atom of radius r and the volume per atom is $8r^3$. This gives

$$8r^3 = 1.66 \times 10^{-29}$$

and hence $r = 1.28 \times 10^{-10}\,\text{m} = 1.28\,\text{Å}$. It is possible to pack spheres much more efficiently than in a simple cubic lattice (the packing in Al is actually face-centred cubic — the most efficient regular packing of spheres), and so the true answer must lie somewhere between these two estimates.

2. The KE of the electrons is $25\,\text{keV} = 0.025\,\text{MeV}$. Since the KE is the total energy, $m\gamma c^2$, minus the rest energy, mc^2, this gives:

$$0.025 = \gamma mc^2 - mc^2 = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) \times 0.51\ .$$

> Aside: I remember finding these "relativistic" units very confusing the first time I met them. The statement that the mass m of an electron is $0.51\,\text{MeV}/c^2$ is exactly equivalent to the statement that mc^2 (which is, of course, an energy) is equal to $0.51\,\text{MeV}$. Similarly, if you are told that the momentum p is equal to $1\,\text{MeV}/c$, the implication is that pc (which also has the dimensions of an energy) is equal to $1\,\text{MeV}$.

A little algebra now gives

$$1 - \frac{v^2}{c^2} = \frac{1}{\left(1 + \frac{0.025}{0.51}\right)^2}\ ,$$

and hence

$$\frac{v}{c} \approx 0.30\ .$$

The relativistic effects (which depend on v^2/c^2) are small but not negligible. Television designers presumably need to take them into account.

3. For a very rough estimate, equate $k_B T$ (or $3k_B T/2$ if you insist) to 13.6 eV. This gives:

$$T = \frac{13.6 \times 1.6 \times 10^{-19}}{1.38 \times 10^{-23}} \approx 160,000\,\mathrm{K}\ .$$

Travelling Waves

4. To work out the wavelength, note that

$$\begin{aligned} \psi\left(x + \frac{2\pi}{k}, t\right) &= a\cos\left(-k\left[x + \frac{2\pi}{k}\right] - \omega t + \phi\right) \\ &= a\cos(-kx - \omega t + \phi - 2\pi) \;=\; \psi(x,t)\ . \end{aligned}$$

Hence, $\psi(x,t)$ changes by one full period as x increases by $2\pi/k$ at constant t. In other words, the wavelength $\lambda = 2\pi/k$.

Similarly, to work out the time period, note that

$$\begin{aligned} \psi\left(x, t + \frac{2\pi}{\omega}\right) &= a\cos\left(-kx - \omega\left[t + \frac{2\pi}{\omega}\right] + \phi\right) \\ &= a\cos(-kx - \omega t + \phi - 2\pi) \;=\; \psi(x,t)\ . \end{aligned}$$

Hence, $\psi(x,t)$ changes by one full period as t increases by $2\pi/\omega$ at constant x. In other words, the time period $T = 2\pi/\omega$. The frequency $\nu = 1/T$ is therefore given by $\nu = \omega/2\pi$.

To work out the velocity of the crests, consider the crest at the point x_{crest} where the argument of the cosine function is zero:

$$-kx_{\mathrm{crest}} - \omega t + \phi = 0\ .$$

This equation rearranges to

$$x_{\mathrm{crest}} = \frac{\phi}{k} - \frac{\omega}{k}t\ .$$

Comparing with the equation $x_{\mathrm{crest}} = x_0 + vt$ that describes uniform motion at velocity v, we see that the wave crest is at position $x_0 = \phi/k$ at time $t = 0$ and is moving at velocity $-\omega/k$.

5. The phase velocity is

$$v = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} \approx \sqrt{\frac{9.8 \times 10}{2\pi}} = 3.96\,\mathrm{ms}^{-1}\ .$$

Complex Representation of Waves, Interference and Diffraction

6. Since $e^{i\theta} = \cos\theta + i\sin\theta$, it follows that $\cos\theta = \mathrm{Re}(e^{i\theta})$ and $\sin\theta = \mathrm{Re}(-ie^{i\theta})$. Hence

$$\cos\theta + \sin\theta \;=\; \mathrm{Re}\left(e^{i\theta} - ie^{i\theta}\right) \;=\; \mathrm{Re}\left((1-i)e^{i\theta}\right)\ .$$

The prefactor $1-i$ may be rewritten in the form $re^{i\phi}$, where:

$$\begin{aligned} r &= \sqrt{[\mathrm{Re}(1-i)]^2 + [\mathrm{Im}(1-i)]^2} = \sqrt{1^2+(-1)^2} = \sqrt{2}\,, \\ \phi &= \tan^{-1}\left(\frac{\mathrm{Im}(1-i)}{\mathrm{Re}(1-i)}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}\,. \end{aligned}$$

(An alternative way to find r and ϕ is to draw the complex number $1-i$ in the Argand diagram and note its length and argument.)

Hence

$$\begin{aligned} \cos\theta + \sin\theta &= \mathrm{Re}\left(\sqrt{2}e^{-i\pi/4}e^{i\theta}\right) = \sqrt{2}\mathrm{Re}\left(e^{i(\theta-\pi/4)}\right) \\ &= \sqrt{2}\cos(\theta-\pi/4)\,, \end{aligned}$$

as required.

7. (i) Using real arithmetic only:

$$\begin{aligned} \psi(x,t) &= a\cos(kx-\omega t) + a\cos(-kx-\omega t) \\ &= a(\cos kx\cos\omega t + \sin kx \sin\omega t) + a(\cos kx\cos\omega t - \sin kx\sin\omega t) \\ &= 2a\cos kx\cos\omega t\,, \end{aligned}$$

where the first step used the trigonometic identity

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi,$$

and the information that $\cos\theta$ is an even function $[\cos(-\theta) = \cos\theta]$ while $\sin\theta$ is an odd function $[\sin(-\theta) = -\sin\theta]$.

(ii) Using complex arithmetic:

$$\psi(x,t) = ae^{i(kx-\omega t)} + ae^{i(-kx-\omega t)}\,.$$

(Actually, of course, $\psi(x,t)$ is the real part of the expression on the RHS. From now on we shall take this as understood.) Hence:

$$\psi(x,t) = ae^{-i\omega t}\left(e^{ikx}+e^{-ikx}\right) = 2ae^{-i\omega t}\cos kx\,.$$

This step used the identities

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta\,,$$

from which it follows that

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \quad (\text{and} \quad e^{i\theta} - e^{-i\theta} = 2i\sin\theta)\,.$$

Taking the real part of $\psi(x,t)$ now gives the real wavefunction

$$\psi(x,t) = 2a\cos kx\cos\omega t\,,$$

exactly as in part (i).

The amplitude a_{total} of $\psi(x,t)$ depends on position,

$$a_{\text{total}}(x) = 2a\cos kx \ ,$$

and hence so does the intensity,

$$I_{\text{total}}(x) = a_{\text{total}}^2(x) = 4a^2\cos^2 kx \ .$$

The position average of the intensity is $4a^2$ times the position average of $\cos^2 kx$. As long as you average over a whole number of half periods, the average value of $\cos^2\theta$ (or $\sin^2\theta$) is equal to 1/2 (this is easy to prove by starting from the trigonometic identity $\cos^2\theta = (1+\cos(2\theta))/2$ and noting that $\cos(2\theta)$ averages to zero). Hence, the average intensity is $2a^2$.

This is expected because the average intensity is proportional to the average energy per unit volume. Assuming that energy is conserved, the average energy density of the standing wave must equal the sum of the average energy densities of the two travelling waves. Both travelling waves have intensity a^2 (independent of position), and hence the average intensity of the standing wave must be $2a^2$.

8. (i) The wave emerging from the segment Δy at height y is

$$Ae^{i(k[\zeta-(-y\sin\theta)]-\omega t)}\Delta y \quad = \quad Ae^{i(k\zeta-\omega t+ky\sin\theta)}\Delta y \ .$$

(ii) The total wave emerging in the ζ direction is the sum of the waves emerging from all the little segments:

$$\psi(\zeta,t) \quad = \quad \sum_{\text{segments}} Ae^{i(k\zeta-\omega t+ky\sin\theta)}\Delta y \ ,$$

where, for each segment (or, equivalently, for each term in the sum), y is the position of the centre of that segment. In the limit as $\Delta y \to 0$, the sum turns into the integral:

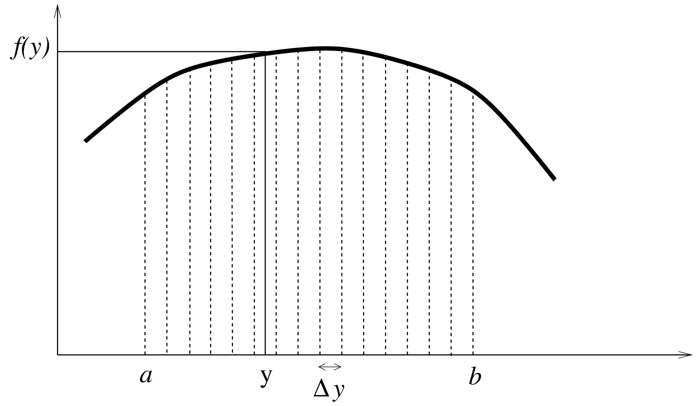
$$\psi(\zeta,t) \quad = \quad \int_{-d/2}^{d/2} Ae^{i(k\zeta-\omega t+ky\sin\theta)}dy \ .$$

Since $e^{i(k\zeta-\omega t+ky\sin\theta)} = e^{i(k\zeta-\omega t)}e^{iky\sin\theta}$, this is equivalent to

$$\psi(\zeta,t) \quad = \quad Ae^{i(k\zeta-\omega t)}\int_{-d/2}^{d/2} e^{iky\sin\theta}dy \ ,$$

as required.

If you are confused about the relationship between the sum and the integral, consider the diagram below. The integral of $f(y)$ from a to b is the sum of the areas of the segments of width Δy.



Since the area of each segment is approximately $f(y)\Delta y$, it follows that

$$\int_a^b f(y)dy \approx \sum_{\text{segments}} f(y)\Delta y \; .$$

In fact, the integral is normally *defined* as the limit of this sum as Δy tends to zero.

(iii) The integral

$$\int_{-d/2}^{d/2} e^{iky\sin\theta} dy$$

may be written as

$$\int_{-d/2}^{d/2} e^{\alpha y} dy \; ,$$

where $\alpha = ik\sin\theta$. This is easy to integrate:

$$\begin{aligned}\int_{-d/2}^{d/2} e^{\alpha y} dy &= \left[\frac{e^{\alpha y}}{\alpha}\right]_{-d/2}^{d/2} = \frac{1}{\alpha}\left(e^{\alpha d/2} - e^{-\alpha d/2}\right) \\ &= \frac{e^{\frac{1}{2}ikd\sin\theta} - e^{-\frac{1}{2}ikd\sin\theta}}{ik\sin\theta} = \frac{d\sin\left(\frac{kd\sin\theta}{2}\right)}{\frac{kd\sin\theta}{2}} \; ,\end{aligned}$$

where the last step used the identity $e^{i\phi} - e^{-i\phi} = 2i\sin\phi$ discussed in the answer to question 7.

The wavefunction $\psi(\zeta, t)$ is therefore given by:

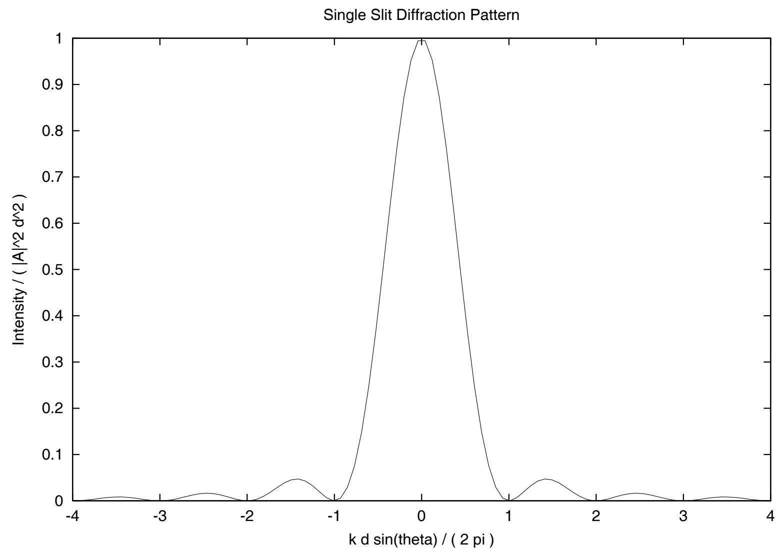
$$\psi(\zeta, t) = Ad\frac{\sin\left(\frac{kd\sin\theta}{2}\right)}{\frac{kd\sin\theta}{2}} e^{i(k\zeta - \omega t)} \; .$$

The intensity I is the square modulus of the complex amplitude (which is everything in front of the $e^{i(k\zeta-\omega t)}$ factor). Hence

$$I = \frac{|A|^2 d^2 \sin^2\left(\frac{kd\sin\theta}{2}\right)}{\left(\frac{kd\sin\theta}{2}\right)^2} \; ,$$

as required.

(iv) The diffraction pattern looks like this (note that the horizontal axis shows values of $\frac{kd\sin\theta}{2\pi}$ instead of values of $\frac{kd\sin\theta}{2}$):



The first zero occurs where $\frac{kd\sin\theta}{2\pi} = 1$ and hence where

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{d\sin\theta} .$$

This implies that

$$\lambda = d\sin\theta .$$

The difference in the lengths of the paths emerging from the top and bottom of the slit is one wavelength.

Photons

9. The total energy entering each eye per second is the energy striking a unit area per second times the area of the pupil:

$$\begin{aligned} \text{energy entering eye per second} &= 1.4 \times 10^{-10} \times \pi(0.0035)^2 \\ &\approx 5.39 \times 10^{-15}\ \text{J} . \end{aligned}$$

Average number of photons entering eye per second is

$$\begin{aligned} \frac{\text{energy entering eye per second}}{\text{energy per photon}} &= \frac{5.39 \times 10^{-15}}{hc/\lambda} \\ &= \frac{5.39 \times 10^{-15} \times 500 \times 10^{-9}}{6.63 \times 10^{-34} \times 3.00 \times 10^{8}} \\ &\approx 13,500 . \end{aligned}$$

The average number of photons inside eye at any one time is

$$\frac{\text{number entering per second} \times \text{length of eye}}{\text{distance a photon travels per second}}$$
$$= \frac{13500 \times 0.04}{3.00 \times 10^8} \approx 1.8 \times 10^{-6} .$$

The actual number of photons in the eye is almost always zero.

Since light always arrives as individual photons, all light detectors must be capable of detecting individual photons. A more interesting question is whether the arrival of a single photon is sufficient to trigger one of the detectors (rods and cones) in the retina, or whether it is necessary to bombard that detector with many photons in close succession. Given that the eye takes much less than 1 s to process a new image, it is reasonable to assume that the "memory" of the detectors is less than, say, 0.1 s. Any effects caused by photons that arrived more than 0.1 s ago can therefore be ignored. Within 0.1 s, only 1,350 photons enter the eye, all of which are focused onto the small area of the retina where the image is formed. Are there more than 1,350 detectors in this area? I have no idea, but I doubt it. In other words, the ability of the eye to see the star provides no convincing evidence that the detectors in the retina are triggered by single photons — it may be necessary to hit the same detector with several photons in quick succession.

10. The electron energy is 30 keV and so the energy $h\nu = hc/\lambda$ of the X-ray photons produced must be less than or equal to 30 keV. The minimum photon wavelength is therefore

$$\lambda = \frac{hc}{30\,\text{keV}} = \frac{6.63 \times 10^{-34} \times 3.00 \times 10^8}{30 \times 10^3 \times 1.6 \times 10^{-19}} = 4.14 \times 10^{-11}\,\text{m}.$$

11. (i) Light of wavelength greater than $\lambda_{\text{max}} = 310\,\text{nm}$ is incapable of producing a current. Hence the work function W is given by:

$$W = \frac{hc}{\lambda_{\text{max}}} = \frac{6.63 \times 10^{-34} \times 3.00 \times 10^8}{310 \times 10^{-9}} \approx 6.42 \times 10^{-19}\,\text{J} .$$

$$W = \frac{6.42 \times 10^{-19}}{1.60 \times 10^{-19}} \approx 4.00\,\text{eV} .$$

(ii) The energy of a photon of wavelength 200 nm is

$$E = \frac{hc}{\lambda} = \frac{6.63 \times 10^{-34} \times 3.00 \times 10^8}{200 \times 10^{-9}} \approx 9.95 \times 10^{-19}\,\text{J} ,$$

or

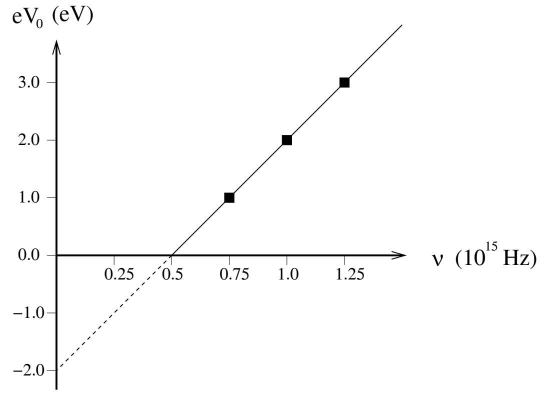
$$E = \frac{9.95 \times 10^{-19}}{1.60 \times 10^{-19}} \approx 6.22\,\text{eV} .$$

(iii) The stopping potential V_0 at 200 nm is given by Einstein's equation, $W + eV_0 = E$. Hence

$$V_0 = 6.22 - 4.00 = 2.22\,\text{V} .$$

The maximum KE of the emitted electrons is 2.22 eV.

12. (i) The data look like this:



Since the y intercept is $-W$, the work function W is 2.0 eV.

(ii) The slope of the line is

$$\frac{(3.0 - 1.0)\,\text{eV}}{(1.25 - 0.75) \times 10^{15}\,\text{Hz}} = \frac{2 \times 1.60 \times 10^{-19}\,\text{J}}{0.5 \times 10^{15}\,\text{s}^{-1}} = 6.4 \times 10^{-34}\,\text{Js} .$$

Assuming that the experimental errors were reflected in the precision with which the measured values were quoted, this is consistent with the accepted value of 6.63×10^{-34} Js. (The fact that the three data points lie on a perfect straight line is suspicious, though.)