Answer to Quantum Physics Classwork 1 February 22^{nd} 2008

Why Represent Wave with Complex Numbers?

Real Version

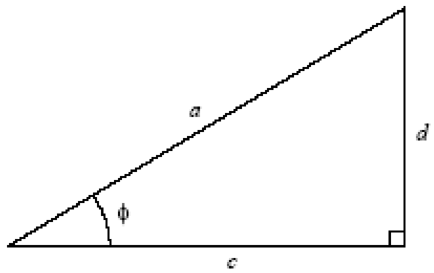
1. $$\begin{aligned} \psi(x,t) &= a_1 \cos(kx - \omega t + \phi_1) + a_2 \cos(kx - \omega t + \phi_2) \\ &= a_1 [\cos(kx - \omega t)\cos\phi_1 - \sin(kx - \omega t)\sin\phi_1] \\ &\qquad + a_2 [\cos(kx - \omega t)\cos\phi_2 - \sin(kx - \omega t)\sin\phi_2] \\ &= (a_1 \cos\phi_1 + a_2 \cos\phi_2)\cos(kx - \omega t) \\ &\qquad - (a_1 \sin\phi_1 + a_2 \sin\phi_2)\sin(kx - \omega t) \\ &= c\cos(kx - \omega t) - d\sin(kx - \omega t)\ , \end{aligned}$$

   where

   $$c = a_1 \cos\phi_1 + a_2 \cos\phi_2 \qquad \text{and} \qquad d = a_1 \sin\phi_1 + a_2 \sin\phi_2$$

   as required.

2. From the right-angled triangle



   we see that $c = a\cos\phi$ and $d = a\sin\phi$, where

   $$a = \sqrt{c^2 + d^2} \quad \text{and} \quad \phi = \tan^{-1}(d/c)\ .$$

   Hence

   $$\begin{aligned} \psi(x,t) &= c\cos(kx - \omega t) - d\sin(kx - \omega t) \\ &= a[\cos\phi\cos(kx - \omega t) - \sin\phi\sin(kx - \omega t)]\ , \end{aligned}$$

   as required.

3. Using the identity $\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 = \cos(\theta_1 + \theta_2)$, the above expression for ψ becomes

$$\psi(x,t) = a\cos(kx - \omega t + \phi) \ .$$

From the triangle that defines a we have $a^2 = c^2 + d^2$. Combining this with the expressions for c and d in terms of a_1, a_2, ϕ_1 and ϕ_2 gives

$$\begin{aligned} a^2 &= (a_1\cos\phi_1 + a_2\cos\phi_2)^2 + (a_1\sin\phi_1 + a_2\sin\phi_2)^2 \\ &= a_1^2(\cos^2\phi_1 + \sin^2\phi_1) + a_2^2(\cos^2\phi_2 + \sin^2\phi_2) \\ &\quad + 2a_1a_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) \\ &= a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2) \ , \end{aligned}$$

where the last step used the result $\cos(\phi_1 - \phi_2) = \cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2$ given in the classwork. Hence

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2)} \ ,$$

as required.

The triangle also shows that $\tan\phi = d/c$. Combining this with the expressions for c and d in terms of a_1, a_2, ϕ_1 and ϕ_2 gives

$$\phi = \tan^{-1}\left(\frac{a_1\sin\phi_1 + a_2\sin\phi_2}{a_1\cos\phi_1 + a_2\cos\phi_2}\right) \ ,$$

as required.

Complex Version

4.

$$\begin{aligned} \tilde{\psi}(x,t) &= a_1 e^{i(kx-\omega t+\phi_1)} + a_2 e^{i(kx-\omega t+\phi_2)} \\ &= a_1 e^{i\phi_1} e^{i(kx-\omega t)} + a_2 e^{i\phi_2} e^{i(kx-\omega t)} \\ &= (a_1 e^{i\phi_1} + a_2 e^{i\phi_2}) e^{i(kx-\omega t)} \\ &= A e^{i(kx-\omega t)} \ , \end{aligned}$$

where $A = a_1 e^{i\phi_1} + a_2 e^{i\phi_2}$ as required.

5. Since $a = \sqrt{A^*A}$, we have

$$\begin{aligned} a^2 &= (a_1 e^{-i\phi_1} + a_2 e^{-i\phi_2})(a_1 e^{i\phi_1} + a_2 e^{i\phi_2}) \\ &= a_1^2 + a_2^2 + a_1 a_2 \left(e^{i(\phi_1 - \phi_2)} + e^{-i(\phi_1 - \phi_2)} \right) \\ &= a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_1 - \phi_2) \ , \end{aligned}$$

where the last step used the result $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ given in the classwork.

Since $\phi = \tan^{-1}(\mathrm{Re}(A)/\mathrm{Im}(A))$, we have

$$\begin{aligned} \phi &= \tan^{-1}\left(\frac{\mathrm{Re}(a_1 e^{i\phi_1} + a_2 e^{i\phi_2})}{\mathrm{Im}(a_1 e^{i\phi_1} + a_2 e^{i\phi_2})} \right) \\ &= \tan^{-1}\left(\frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \sin\phi_2} \right) \ , \end{aligned}$$

where the last step used the result $e^{i\theta} = \cos\theta + i\sin\theta$ given in the classwork.

6.

$$\begin{aligned} \psi(x,t) &= \mathrm{Re}(A e^{i(kx - \omega t)}) \\ &= \mathrm{Re}\left(a e^{i\phi} e^{i(kx - \omega t)} \right) \\ &= \mathrm{Re}\left(a e^{i(kx - \omega t + \phi)} \right) \\ &= a\cos(kx - \omega t + \phi) \ , \end{aligned}$$

where the last step used the result $e^{i\theta} = \cos\theta + i\sin\theta$ given in the classwork.