Imperial College London

BSc/MSci EXAMINATION June 2013

This paper is also taken for the relevant Examination for the Associateship

QUANTUM MECHANICS

For 2nd-Year Physics Students

Wednesday, 5th June 2013: 10:00 to 12:00

Answer all questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. Consider a two-dimensional system in which the particle is subject to a potential of the form $V(x, y) = V_x(x) + V_y(y)$.

(i) Show that a solution to the time-independent Schrödinger Equation can be written as $u(x, y) = X(x)Y(y)$ and that each of $X(x)$ and $Y(y)$ satisfies the time-independent Schrödinger Equation of the one-dimensional square well (you do not need to solve the resulting equations).

[5 marks]

The potentials $V_x(x)$ and $V_y(y)$ are each given by one-dimensional square wells of height W and width L_x or L_y. That is,

$$V_x(x) = \begin{cases} 0 & \text{if } |x| \le L_x/2 \\ W & \text{if } |x| > L_x/2 \end{cases}$$

and similar for $V_y(y)$. Consider the case where the *total energy of the particle in two dimensions* is less than W. [For a one-dimensional square well, the eigenstates are u_i with energies E_i when $E_i < W$.]

(ii) Is the particle *bound*, *unbound*, or neither? Explain your reasoning.

[5 marks]

(iii) Would we expect that the energy eigenstates of this system are also eigenstates of angular momentum? Explain your reasoning. [5 marks]

(iv) (a) In terms of the one-dimensional energy eigenvalues E_i, what is the energy of a particle in the state $u(x, y) = u_2(x)u_3(y)$ when $L_x = L_y$?

(b) Again assuming $L_x = L_y$, if the particle has energy $E_i + E_j$, where i and j are integers, $i \neq j$, labelling the one-dimensional eigenstates u_i as above, what is the *most general* eigenstate $u(x, y)$?

[10 marks]

[Total 25 marks]

2. A particle of mass m is confined by an infinite square well of width L centered on the origin.

(i) Show that the function

$$u_2(x) = \sqrt{\frac{2}{L}}\sin(2\pi x/L)$$

is a correctly normalised solution to the time-independent Schrödinger equation for this system. What is the corresponding energy E_2 and the full time-dependent wavefunction $\psi(x,t)$? [5 marks]

We change this system by adding a small perturbation, $e(x)$, to the potential.

(ii) Write down the time-independent Schrödinger Equation for the wavefunction and energy of the perturbed system. Assuming that the perturbation is small, and that therefore the wavefunction can be approximated by the unperturbed solution, show that the energy is shifted by

$$\delta E = \langle e \rangle = \int u^*(x) e(x) u(x)\, dx$$

[10 marks]

The system is perturbed by a harmonic oscillator potential corresponding to a classical angular frequency ω.

(iii) What is the perturbation $e(x)$ for this case? Write down the total quantum mechanical Hamiltonian in terms of the position and momentum operators $\hat{x}$ and $\hat{p}$, and show that it can be written in the form

$$\hat{H} = \hbar\omega(\frac{m\omega}{2\hbar}\hat{x}^2 + \frac{1}{2m\omega\hbar}\hat{p}^2) \qquad -L/2 < x < +L/2\ .$$

[7 marks]

(iv) If the unperturbed particle was in the state $u_2(x)$, given above, show that the perturbation to the energy, δE_2, is proportional to $\hbar\omega\alpha^2 L^2$ and work out the constant of proportionality. [8 marks]

(v) What are the requirements on the physical properties of the system we require so that our use of perturbation theory is valid? [5 marks]

[Total 35 marks]

You may find one or more of the following integrals useful:

$$\int_{-\pi/2}^{+\pi/2} x^2 \sin^2(x)\, dx = \frac{\pi}{24}\left(\pi^2 + 6\right)$$

$$\int_{-\pi}^{+\pi} x^2 \sin^2(x)\, dx = \frac{\pi}{6}\left(2\pi^2 - 3\right)$$

$$\int_{-2\pi}^{+2\pi} x^2 \sin^2(x)\, dx = \frac{8\pi^3}{3} - \pi$$

3. A particle with spin 1/2 is in a state

$$\psi = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

where θ is an arbitrary (real) angle.

(i) Show that this is a normalized state. Is it the most general possible spin state? Why or why not? (Hint: this ψ has only real entries.) [5 marks]

The three-dimensional angular momentum operators $\hat{L}_i$ have the commutation relation

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar \hat{L}_z \, .$$

The *spin* operators $\hat{S}_i$ are given below.

(ii) What is the commutator $\left[\hat{S}_y, \hat{S}_x\right]$ (note the order of the subscripts)? [5 marks]

(iii) The spin of a particle is measured (using a Stern-Gerlach apparatus, say) and found to be in the $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ eigenstate of $\hat{S}_z$.

(a) What is the value of the spin observable, S_z?

(b) What are the probabilities of it subsequently being found in each of the eigenstates of $\hat{S}_x$?

[10 marks]

One general form for the Heisenberg uncertainty principle is given by

$$\Delta Q^2 \, \Delta R^2 \geq \left\langle \frac{i}{2} [\hat{Q}, \hat{R}] \right\rangle^2$$

where ΔQ^2 gives the variance of the distribution of the observable Q.

(iv) (a) Write down the expression for the variance ΔQ^2 in terms of the expectation of Q and Q^2.

(b) Calculate ΔS_x^2 and ΔS_y^2 in the state ψ above, and hence show that

$$\Delta S_x^2 \Delta S_y^2 = \left(\frac{\hbar^2}{4}\right)^2 \cos^2 2\theta \, .$$

[10 marks]

(v) Can we find θ for our state ψ such that it is a minimum-uncertainty state (i.e., gives an *equality* in the uncertainty relation) with respect to $Q = S_x$ and $R = S_y$? If so, do so. [10 marks]

[This question continues on the next page . . .]

[Total 40 marks]

The spin operators are $\hat{S}_i = \hbar\sigma_i/2$, where the Pauli spin matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

You may use the following facts: the Pauli matrices satisfy $\sigma_i^2 = I$ (the 2×2 identity matrix); the eigenvectors of $\hat{S}_z$ are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$; the eigenvectors of $\hat{S}_x$ are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.