M. Coppins
14.12.07

Problem Sheet 9: Lectures 5.2 and 5.3

The following data are needed for some the questions on this problem sheet.
$G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.
Mass of the Earth $= 5.98 \times 10^{24}$ kg.
Mass of the Sun $= 1.99 \times 10^{30}$ kg.
Radius of the Earth $= 6.37 \times 10^6$ m.
Mean radius of the Earth's orbit about the Sun $= 1.49 \times 10^{11}$ m.
Mean radius of the Moon's orbit about the Earth $= 3.84 \times 10^8$ m.

Exercises

1. If the Earth was a perfectly smooth sphere (no mountains, trees, buildings, people, etc) without an atmosphere, an object could orbit just above the ground. Calculate the speed of such an object.

2. Show that a satellite in a circular orbit around a planet of mass M has an orbital period given by $T = \dfrac{2\pi r^{3/2}}{\sqrt{GM}}$, where r is the radius of the orbit.

3. Calculate the radius of a geostationary orbit around the Earth.

4. Uranus has a moon Oberon which has an orbital period of 1.16×10^6 s and a mean orbital radius of 5.84×10^8 m. Calculate the mass of Uranus.

Problems

5. (i) The Moon experiences gravitational forces from both the Earth and the Sun. Calculate the ratio of the magnitude of these two forces on the moon.

   (ii) You should have found that the Sun exerts a larger force than the Earth. So, why isn't the Moon in orbit around the Sun?

Continued overleaf

6. (i) In Problem Sheet 6 (Q. 5) we considered the Bohr model of the Hydrogen atom, in which an electron follows a circular orbit around a proton. We found that the speed of the electron is given by $v = e/\sqrt{4\pi\epsilon_0 m_e r}$, where r is the radius of the orbit, and m_e is the electron mass. Given that the electrical potential energy of an electron distance r from a proton is $U = -\dfrac{e^2}{4\pi\epsilon_0 r}$, show that $|E| = |K| = \dfrac{1}{2}|U|$, where E and K are the electron's total mechanical energy and kinetic energy respectively. [We derived this relationship for the case of a circular gravitational orbit in Lecture 5.2. It applies in both cases because both the electrical force and gravity are inverse square ($\propto r^{-2}$) central forces.]

   (ii) We found that the quantization of the electron's orbital angular momentum meant that only certain orbits were allowed, characterized by the principal quantum number, n. The radius of the n^{th} orbit is

$$r_n = n^2 \frac{\hbar^2 4\pi\epsilon_0}{m_e e^2} \qquad \text{[see Problem Sheet 6, Q. 5.]}$$

   Calculate the total mechanical energy of an electron in the ground state ($n = 1$).

   (iii) Hence find the ionization energy of Hydrogen (i.e., the energy required to remove the electron from the ground state to $r = \infty$). Give your answer in eV.

   $\hbar = 1.05 \times 10^{-34}$ kg m^2 s^{-1}

   $\epsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}

   $m_e = 9.11 \times 10^{-31}$ kg

   $e = 1.60 \times 10^{-19}$ C

   1 eV $= 1.60 \times 10^{-19}$ J

7. In Lecture 5.2 we saw that energy conservation for an object of mass m in orbit around an object of mass M looks like the equation for an object executing 1-D (in the radial direction) motion in a "pseudo-potential" $U^* = \dfrac{L^2}{2mr^2} - \dfrac{GMm}{r}$.

   (i) Show that the minimum of U^* occurs at a radius given by $r_c = \dfrac{L^2}{GMm^2}$ and that the minimum value is given by $U^*_{min} = -\dfrac{GMm}{2r_c}$.

   (ii) Qualitatively describe the objects motion if its total mechanical energy is: (a) equal to U^*_{min}, (b) between U^*_{min} and zero, (c) greater than zero.

8. In Classwork I we found that the tidal force acting on a composite object consisting of two spheres (each of mass m and radius a) falling under gravity towards a black hole (of mass M) is given by $F_{tidal} = \dfrac{4GMma}{r^3}$, where r is the distance of the composite body from the black hole, and we have assumed $r >> a$. If the composite body was in orbit around the black hole it would still be constantly falling, so the expreesion for F_{tidal} would still apply. And, of course, any object will give rise to tidal forces, not just a black hole.

   (i) Consider a composite object in orbit around a planet of mass M. Show that if the composite object is held together by gravity then it cannot exist below a minimum distance from the centre of the planet, given by $r_{min} = a(16M/m)^{1/3}$. (This minimum distance is called the Roche limit.)

   (ii) Assuming that the planet and the composite body have equal densities, find the ratio of r_{min} to the radius of the planet.

   (iii) Artificial satellites orbit the Earth well within the Roche limit. Why aren't they torn apart?

Numerical Answers

1. 7.91×10^3 m s^{-1}

3. 4.22×10^7 m

4. 8.76×10^{25} kg

5. (i) $F_{from\ Sun}/F_{from\ Earth} = 2.21$

6. (ii) -2.19×10^{-18} J, (iii) $+2.19 \times 10^{-18}$ J or 13.7 eV [Using more accurate values of the constants gives 13.6 eV]

8. (ii) 2.52.