MECHANICS PROBLEM SHEET 8, ANSWERS

1. Uniform sphere: $I = \frac{2}{5} MR^2$

$\therefore L = I\omega = \frac{2}{5} MR^2 \frac{2\pi}{T} = \frac{4\pi}{5} \frac{MR^2}{T}$

Ang mom conserved, M = const $\therefore \frac{R_{init}^2}{T_{init}} = \frac{R_{final}^2}{T_{final}}$

$\therefore T_{final} = T_{init} \left(\frac{R_{fin}}{R_{init}}\right)^2$

$= 25 \times 24 \times 60 \times 60 \times \left(\frac{20}{8.8 \times 10^5}\right)^2 = 1.11 \times 10^{-3}$ s

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2 (i) At surf of planet $U = -\frac{GMm}{R}$ ← mass of object

Min ke needed to reach $r = \infty$ given by $K + U = 0$

$\rightarrow \frac{1}{2} m v_{es}^2 = \frac{GMm}{R} \rightarrow v_{es} = \sqrt{\frac{2GM}{R}}$

(ii) $v_{es} = \left(\frac{2 \times 6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{6.37 \times 10^6}\right)^{\frac{1}{2}} = 1.12 \times 10^4 \, ms^{-1}$

(iii) $v_{es} = \left(\frac{2 \times 6.67 \times 10^{-11} \times 1.99 \times 10^{30}}{1.49 \times 10^{11}}\right)^{\frac{1}{2}} = 4.22 \times 10^4 \, ms^{-1}$

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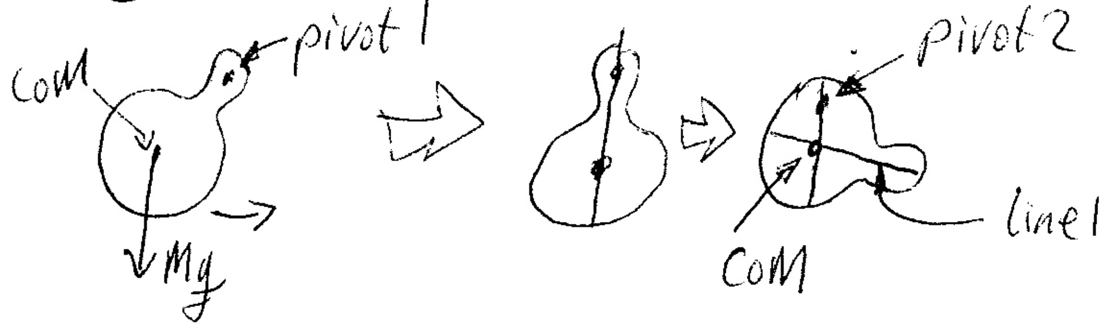
3 The object will hang with CoM vertically below pivot ($\Rightarrow \tau_{grav} = 0$)



4 $\underline{F} = -\frac{d}{dr}\left(-\frac{GmM}{r}\right)\underline{\hat{r}} = GmM\frac{d}{dr}(r^{-1})\underline{\hat{r}} = -\frac{GmM}{r^2}\underline{\hat{r}}$

= force point mass M at centre of sphere.

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5 (i) $I = \text{MoI} = 2 \times m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{2}$

$L = I\omega = \frac{ml^2\omega}{2}$ & $K = \frac{1}{2}I\omega^2 = \frac{ml^2\omega^2}{4}$

(ii) For each mass: $F_c = m\omega^2 r$ — $r = l/2$

But $\omega = \frac{L}{I} = \frac{2L}{ml^2}$ $\therefore F_c = m\frac{4L^2}{m^2l^4}\frac{l}{2} = \frac{2L^2}{ml^3}$

(iii) $K = \frac{ml^2\omega^2}{4} = \frac{ml^2}{4}\frac{4L^2}{m^2l^4} = \frac{L^2}{ml^2}$

$\therefore \Delta K = K_{fin} - K_{init} = \frac{L^2}{m}\left(\frac{1}{l_1^2} - \frac{1}{l_0^2}\right)$

$l_1 < l_0$ $\therefore \Delta K > 0$ i.e. k-e increases

(iv) For one mass

$W_1 = -\int_{r_0}^{r_1} F_c\, dr$ $\quad r_1 = l_1/2$

$F_c = \frac{2L^2}{ml^3}$ $\quad l = 2r$

$r_0 = l_0/2$

$= -\frac{2L^2}{m}\int_{l_0/2}^{l_1/2}\frac{dr}{8r^3} = \frac{-L^2}{4m}\left[-\frac{1}{2r^2}\right]_{l_0/2}^{l_1/2}$

$= \frac{L^2}{8m}\left(\frac{4}{l_1^2} - \frac{4}{l_0^2}\right) = \frac{L^2}{2m}\left(\frac{1}{l_1^2} - \frac{1}{l_0^2}\right)$

Total work done (2 masses) $= \dfrac{L^2}{m}\left(\dfrac{1}{l_1^2} - \dfrac{1}{l_0^2}\right)$

$= \Delta k$ [part (iii)] i.e. WET OK

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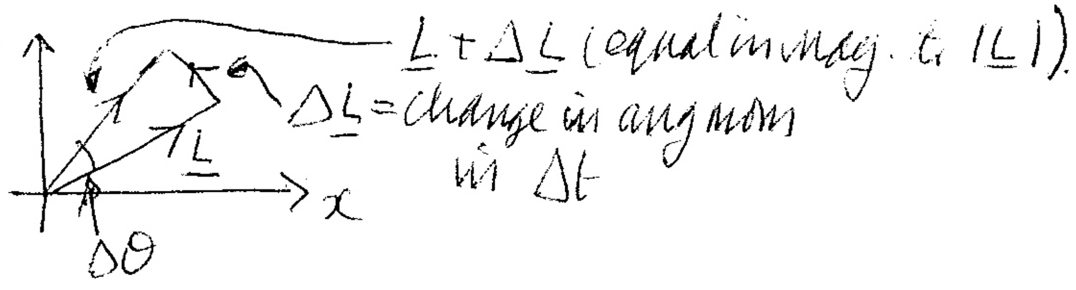
6 (i)



From diagram: $\Delta L = L\Delta\theta = I\omega\Delta\theta$ But $\Delta L = \tau\Delta t$

$\therefore \Delta\theta = \dfrac{\tau\Delta t}{I\omega}$ $\therefore \Omega = \dfrac{d\theta}{dt} = \dfrac{\Delta\theta}{\Delta t} = \dfrac{\tau}{I\omega}$ ← rMg

(ii) As ω decreases Ω increases i.e. the gyroscope precesses faster.

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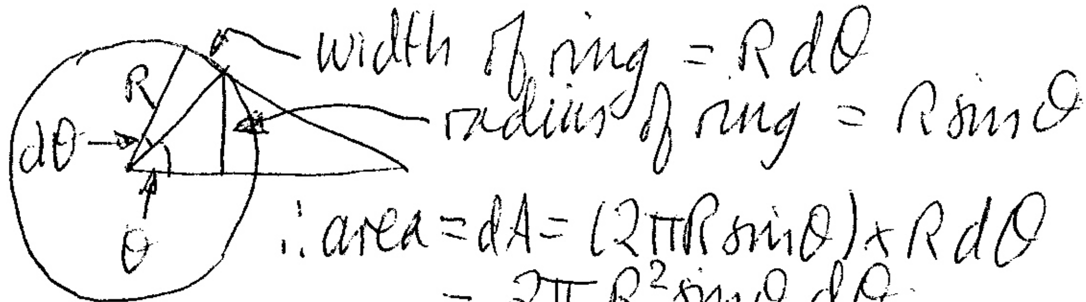
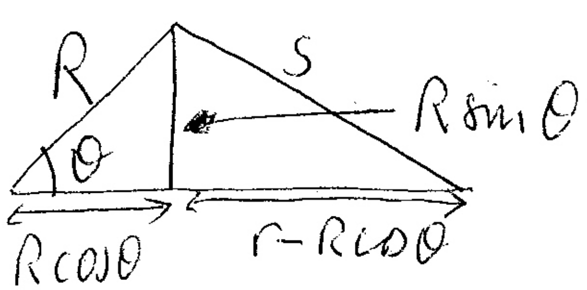
7 (i)



$\therefore$ area $= dA = (2\pi R\sin\theta) \times R\,d\theta$
$= 2\pi R^2 \sin\theta\, d\theta$

(ii)



Pythag: $S^2 = (r - R\cos\theta)^2 + R^2\sin^2\theta$
$= r^2 - 2rR\cos\theta + \underbrace{R^2\cos^2\theta + R^2\sin^2\theta}_{= R^2}$

$$\therefore \frac{d}{d\theta}(s^2) = \frac{d}{d\theta}(r^2 - 2rR\cos\theta + R^2)$$

$$LHS = 2s\frac{ds}{d\theta}, \quad RHS = +2rR\sin\theta$$

$$\therefore s\frac{ds}{d\theta} = rR\sin\theta \quad \therefore \sin\theta\, d\theta = \frac{s\,ds}{rR}$$

(iii) $\frac{dm}{m_s} = \frac{dA}{4\pi R^2}$ ← $2\pi R^2 \underbrace{\sin\theta\, d\theta}$ $\frac{s\,ds}{rR}$

area of shell

$$\therefore dm = m_s \frac{2\pi R^2}{4\pi R^2}\frac{s\,ds}{rR} = \frac{m_s\, s\,ds}{2rR}$$

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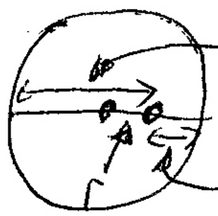
8  

$S_{max} = r + R$

$S_{min} = R - r$

$$U = \int_{s=R-r}^{s=R+r}\left(-\frac{Gmm_s\,ds}{2rR}\right) = -\frac{Gmm_s}{2rR}\underbrace{[s]_{R-r}^{R+r}}_{R+r-(R-r)=2r}$$

$\therefore U = -\frac{Gmm_s}{R}$ = uniform (same everywhere inside shell)

Any hollow sphere is made up of concentric shells. Adding the contributions from each shell
→ U = uniform (i.e. ind of r) inside
→ $F = -\frac{dU}{dr} = 0$ → no gravitational force inside hollow sphere

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