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First Year Physics 7.12.07

Problem Sheet 8: Lectures 4.4–5.1

Exercises

1. At the end of their lives stars somewhat larger than our Sun can collapse to form an ultra-high density neutron star. Consider such a star which before the collapse had a radius of 8.8×10^5 km, and a rotation period of 25 days. Calculate its rotation period if it collapses to form a neutron star of radius 20 km. You may assume that both the original star and the neutron star are uniform spheres.

2. (i) Consider an object on the surface of a spherical planet. The planet has mass M and radius R. If the object has enough kinetic energy it can completely escape from the planet's gravity (i.e., reach $r = \infty$). The minimum speed which the object must have to achieve this is called the escape velocity. Show that it is given by $v_{es} = \sqrt{2GM/R}$.

   (ii) What is the escape velocity from the Earth?

   (iii) What is the escape velocity of the Earth from the Sun?
   [Mass of the Earth = 5.98×10^{24} kg.
   Mass of the Sun = 1.99×10^{30} kg.
   Radius of the Earth = 6.37×10^6 m.
   Mean radius of the Earth's orbit about the Sun = 1.49×10^{11} m.]

3. The centre of mass of a flat object can be determined by pivoting it about one point, allowing it to come to rest, drawing a vertical line on it through the pivot, pivoting it from another point, and drawing a second vertical line. The centre of mass is located at the intersection of the lines. Why does this method work?

4. The gravitational potential energy of a particle of mass m outside a spherically symmetric mass distribution (total mass M) and distance r from its centre is $U = -\dfrac{GmM}{r}$. Use the equation $\mathbf{F} = -\dfrac{dU}{dr}\hat{\mathbf{r}}$ to show that the gravitational force on the particle is identical to the force that would be exerted on it by a point mass M at the centre of the sphere.

Problems

5. When a spinning ice-skater pulls in his arms he spins faster. This is a manifestation of the conservation of angular momentum. We can get some insight into this effect by analyzing a simpler system, consisting of a two point masses, each of mass m, held some distance apart by a massless rod, and spinning about an axis through the mid-point of the rod with an angular speed ω.

   (i) Assuming that the separation of the masses is l, show that the total angular momentum of the system is $L = \dfrac{ml^2\omega}{2}$ and its total kinetic energy is $K = \dfrac{ml^2\omega^2}{4}$.

Continued overleaf

(ii) Each mass experiences a centripetal acceleration as it moves in a circle around the axis. The centripetal force is due to the tension in the massless rod. Show that the centripetal force on one mass is $F_c = \frac{2L^2}{ml^3}$, where L is the total angular momentum of the two-mass system.

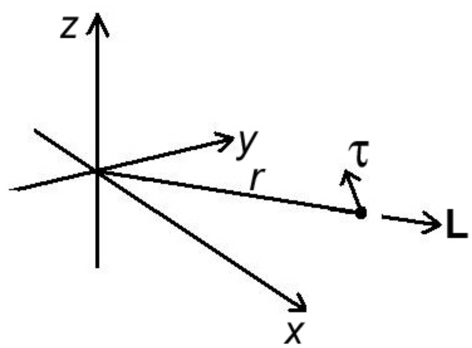
(iii) The separation is reduced from l_0 to l_1. The angular momentum L is constant in this process. Show that the change in kinetic energy is

$$\Delta K = \frac{L^2}{m}\left(\frac{1}{l_1^2} - \frac{1}{l_0^2}\right) .$$

Does the kinetic energy increase or decrease?

(iv) The work done by a radially inward force of magnitude F on a particle whose distance from the origin changes from r_0 to r_1, is given by $W = -\int_{r_0}^{r_1} F dr$. Find an expression for the work done on each mass by the centripetal force while the separation is being reduced, and, hence, show that the work-energy theorem is satisfied.

6. In Lecture 4.4 we found that a spinning gyroscope precesses. At any instant the flywheel experiences a torque τ, the direction of which lies in the x-y (horizontal) plane (see figure, below) .



(i) Show that in a short time Δt the gyroscope precesses through an angle in the x-y plane given by:

$$\Delta\theta = \frac{\tau\delta t}{I\omega}$$

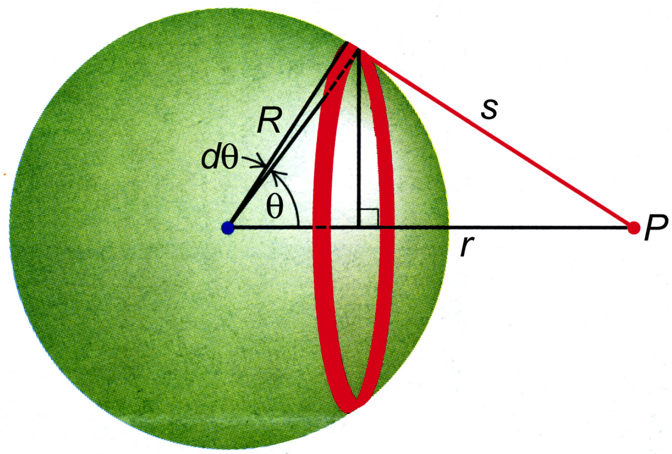
where I is the moment of inertia of the flywheel about its centre of mass, and ω is its angular speed about the axis through the centre of mass. Hence show that the precessional angular speed is given by:

$$\Omega = \frac{d\theta}{dt} = \frac{rMg}{I\omega}$$

where M is the mass of the flywheel, and r is the distance of the its centre of mass from the pivot.

(ii) The flywheel will gradually slow down due to friction. What effect will this have on the precession?

7. (i) Consider a uniform spherical shell of mass m_s. The figure (below) shows a ring on this shell, the points of which are all distance s from point P, which itself is distance r from the centre of the shell. Show that the surface area of the ring is $dA = 2\pi R^2 \sin\theta d\theta$, where $d\theta$ is the angle subtended by the width of the ring at the centre of the sphere.



(ii) Show that $s^2 = r^2 - 2rR\cos\theta + R^2$ and, hence, by differentiating this equation, that $\sin\theta d\theta = \dfrac{sds}{rR}$.

(iii) Since the spherical shell is uniform the ratio of the mass of the ring, dm, to the mass of the whole shell, m_s, is just equal to the ratio of their areas. Show that

$$dm = \frac{m_s sds}{2rR} .$$

[This equation was used in Lecture 5.1.]

8. Consider a mass m *inside* a uniform spherical shell, distance r from its centre. The mass of the shell is m_s and its radius is R. The potential energy can again be found by summing over contributions from rings on the shell. For any one ring we find $dU = -\dfrac{Gmm_s ds}{2rR}$, just as we did when the mass was outside the shell (Lecture 5.1). In fact, the only difference from the latter situation is in the choice of limits when integrating over s. Determine the appropriate limits, and show that the potential energy is

$$U = -\frac{Gmm_s}{R} .$$

What does this tell us about the gravitational force inside a hollow sphere?

Numerical Answers

1. 1.11×10^{-3} s.

2. (ii) 1.12×10^{4} m s^{-1}, (iii) 4.22×10^{4} m s^{-1}.