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First Year Physics | 11.12.07

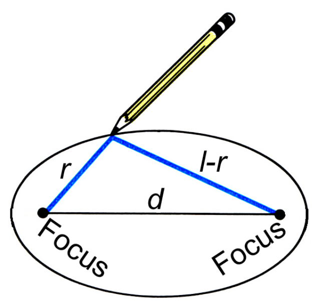
Handout 4: Geometrical properties of ellipses and Kepler's laws of planetary motion

1. Introduction

Newton's explanation of Kepler's empirically determined laws of planetary motion is one of the great landmarks in the history of human culture, and anyone studying classical mechanics should have some insight into it. This handout covers the required background material on the geometry of ellipses, which is used in Lecture 5.2 and the Supplementary Problem Sheet. You are not expected to memorize any of this material.

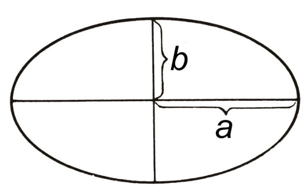
2. Basic properties of ellipses

Fix the ends of a piece of string (length l) distance d apart, and keeping the string taut trace out an ellipse.



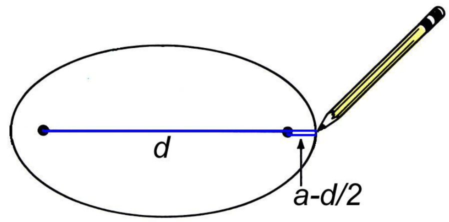
The fixed ends of the string are the *foci*. The eccentricity is defined as:

$$e = \frac{d}{l} \tag{1}$$



a = semi-major axis, b = semi-minor axis.

When the pencil is at end of the major axis we see that:



$$l = d + 2\left(a - \frac{d}{2}\right)$$

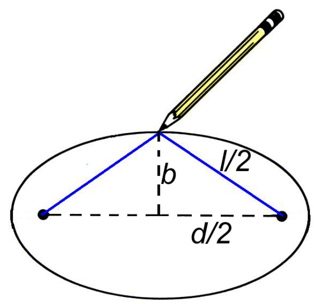
i.e.,

$$l = 2a \tag{2}$$

Using Eq. 1 we therefore find

$$d = 2ea\ . \tag{3}$$

When the pencil is at the end of minor axis we see that:



$$\frac{l^2}{4} = b^2 + \frac{d^2}{4}$$

i.e.,

$$b^2 = \frac{l^2}{4}\left(1 - \frac{d^2}{l^2}\right)$$

Substituting from Eq's 1 and 2 we find:

$$b^2 = a^2(1 - e^2) \tag{4}$$

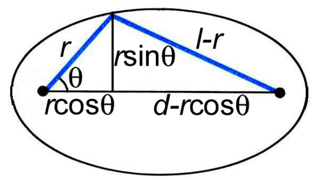
or

$$e^2 = 1 - \frac{b^2}{a^2} \tag{5}$$

The limiting values of the eccentricity are:
•$e_{max} = 1$ (ellipse reduces to straight line, $b = 0$).
•$e_{min} = 0$ (ellipse reduces to circle, $a = b = r$).

3. Equation of ellipse in polar coordinates

Use polar coordinates with origin at one focus.



$$(l-r)^2 = (r\sin\theta)^2 + (d - r\cos\theta)^2$$

Expanding

$$l^2 - 2lr + r^2 = r^2\sin^2\theta + d^2 - 2dr\cos\theta + r^2\cos^2\theta\ ,$$

and using $\sin^2\theta + \cos^2\theta = 1$ we find

$$l^2 - 2lr = d^2 - 2dr\cos\theta\ .$$

Substituting from Eq's 2 and 3 gives

$$4a^2 - 4ar = 4e^2a^2 - 4ear\cos\theta$$

which reduces to

$$r(1 - e\cos\theta) = a(1 - e^2)$$

i.e.,

$$r = \frac{a(1-e^2)}{(1 - e\cos\theta)} \tag{6}$$

or, substituting from Eq. 4,

$$r = \frac{b^2}{a(1 - e\cos\theta)} \tag{7}$$

Equations 6 and 7 are alternative forms of the equation of an ellipse in polar coordinates.

4. Area of ellipse

The element subtended by angle $d\theta$ at origin has area

$$dA = \frac{r^2}{2}d\theta$$

(see Lecture 5.2). Thus the total area of the ellipse is given by

$$A = \int_0^{2\pi} \frac{r^2}{2}d\theta$$

Substituting from Eq. 7 we find

$$A = \frac{b^4}{2a^2}\int_0^{2\pi} \frac{d\theta}{(1 - e\cos\theta)^2}$$

which can be evaluated using a standard integral to obtain

$$A = \frac{b^4}{2a^2} \, \frac{2\pi}{(1-e^2)^{3/2}}$$

Substituting from Eq. 4 we therefore find

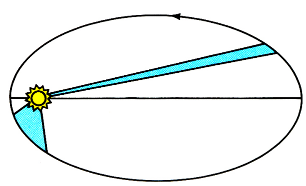
$$A = \frac{2\pi b^4 a^3}{2a^2 b^3} = \pi a b \tag{8}$$

Alternatively we can think of an ellipse as a distorted circle. Start with a circle of radius unity, and area π. Stretch it out in one direction by a factor a, and in the perpendicular direction by a factor b. The resulting shape is an ellipse of semi-major axis a, semi-minor axis b and area πab.

5. Kepler's laws of planetary motion

1. Planets move in elliptical orbits with the sun at one focus.

2. A line from the sun to a planet sweeps out equal areas in equal times.



3. The period of a planets orbit is proportional to $a^{3/2}$, where a is the semi-major axis and the constant of proportionality is the same for all planets.