Classwork VI
An old Exam Question

This classwork consists of one of the long questions from the April 2007 Mechanics and Relativity exam. Each long question is worth 32% of the total mark. Since the exam lasts 2 hours, you should spend approximately 40 minutes on the question. The information needed for the classwork and the numerical answer are printed on the back of the sheet so you can try to do the question under exam conditions.

(i) A planet experiences a gravitational force from a star. Assuming that this is the only force acting on the planet, show that $\mathbf{L}$, its angular momentum about the star, is constant.

[6 marks]

(ii) The planet follows an elliptical orbit around the star. Show that the rate at which a line joining the planet to the star sweeps out area is given by

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

where L is the magnitude of the angular momentum and m is the mass of the planet (i.e., Kepler's second law of planetary motion).

[8 marks]

(iii) Assuming now that the planet actually follows a *circular* orbit of radius R, show that the period of its orbit is given

$$T = \frac{2\pi R^{3/2}}{(GM)^{1/2}}$$

where M is the mass of the star, and G is the gravitational constant.

[7 marks]

(iv) Assuming that the Earth follows a circular orbit around the Sun (mass 1.99×10^{30} kg) of period 1 year, calculate the distance from the Earth to the Sun.

[3 marks]

(v) A binary star system consists of two stars, of masses M_1 and M_2, following circular orbits, of radii r_1 and r_2, about their common centre of mass. Write down Newton's second law for each of the stars separately, and, hence, show that the expression for the period, T, found in part (iii) can be used for the binary star system if R is replaced by the separation of the stars' centres and M is replaced by $M_1 + M_2$.

[8 marks]

[TOTAL 32 marks]

Information needed for this Classwork

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}$$

$L = mv_\theta r$

$v_\theta = \omega r$

Particle in circular orbit of radius r experiences a centripetal acceleration: $\mathbf{a} = -\frac{v^2}{r}\hat{\mathbf{r}}$

Gravitational force between masses m_1 and m_2, distance r apart: $\mathbf{F} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}$

Period of circular orbit: $T = \frac{2\pi r}{v}$

Numerical Answer

(iv) 1.50×10^{11} m