MECHANICS: SUPPLEMENTARY PROBLEM SHEET
ANSWERS

1. $r = \dfrac{\alpha}{(1-e\cos\theta)}$

$\therefore v_r = \dfrac{dr}{dt} = \alpha \dfrac{d}{dt}(1-e\cos\theta)^{-1} = \alpha\left\{-(1-e\cos\theta)^{-2}\right\}\dfrac{d}{d\theta}(-e\cos\theta)$

$= -\dfrac{\alpha}{(1-e\cos\theta)^2}(e\sin\theta)\dfrac{d\theta}{dt}$

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2. $L = mv_\theta r = mr^2\dfrac{d\theta}{dt}$

$\therefore \dfrac{d\theta}{dt} = \dfrac{L}{mr^2} = \dfrac{L}{m}\dfrac{(1-e\cos\theta)^2}{\alpha^2}$

$\therefore v_r = \dfrac{-\alpha e\sin\theta}{(1-e\cos\theta)^2} \times \dfrac{L(1-e\cos\theta)^2}{m\alpha^2} = -\dfrac{Le\sin\theta}{m\alpha}$

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3. Polar equation of ellipse $\rightarrow \cos\theta = \dfrac{1}{e}\left(1-\dfrac{\alpha}{r}\right)$

$\therefore \cos^2\theta = \dfrac{1}{e^2}\left(1-\dfrac{2\alpha}{r}+\dfrac{\alpha^2}{r^2}\right)$

$\therefore \sin^2\theta = 1-\dfrac{1}{e^2}\left(1-\dfrac{2\alpha}{r}+\dfrac{\alpha^2}{r^2}\right)$

$\therefore e^2\sin^2\theta = e^2-1+\dfrac{2\alpha}{r}-\dfrac{\alpha^2}{r^2}$

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4. $e^2\sin^2\theta = 1-\dfrac{b^2}{a^2}-1+\dfrac{2\alpha}{r}-\dfrac{\alpha^2}{r^2} = -\dfrac{b^2}{a^2}+\dfrac{2\alpha}{r}-\dfrac{\alpha^2}{r^2}$

$\therefore v_r^2 = \dfrac{L^2}{m^2\alpha^2}\left(-\dfrac{b^2}{a^2}+\dfrac{2\alpha}{r}-\dfrac{\alpha^2}{r^2}\right)$

$= \dfrac{L^2}{m^2}\left(-\dfrac{b^2}{\alpha^2a^2}+\dfrac{2}{\alpha r}-\dfrac{1}{r^2}\right)$

$$v_\theta = r\frac{d\theta}{dt} = \frac{L}{mr}$$

$$\therefore K = \frac{1}{2}m(v_r^2 + v_\theta^2) = \frac{1}{2}\frac{mL^2}{m^2}\left(\frac{b^2}{a^2 a^2} + \frac{2}{ar} - \frac{1}{r^2} + \frac{1}{r^2}\right)$$

$\frac{a^2}{b^4} \times \frac{b^2}{a^2} = \frac{1}{b^2}$ $\qquad$ $\frac{2a}{b^2 r}$

$$\therefore K = \frac{-L^2}{2mb^2} + \frac{L^2 a}{mb^2 r}$$

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5 $$E + \frac{GMm}{r} = \frac{-L^2}{2mb^2} + \frac{L^2 a}{mb^2 r}$$

Equate const terms: $E = \frac{-L^2}{2mb^2}$

Equate coeffs of r^{-1}: $GMm = \frac{L^2 a}{mb^2} \Rightarrow L^2 = \frac{GMm^2 b^2}{a^2} \Rightarrow L = mb\sqrt{\frac{GM}{a}}$

$$\therefore |E| = \frac{L^2}{2mb^2} = \frac{1}{2mb^2} \times \frac{GMm^2 b^2}{a^2} = \frac{GMm}{2a}$$

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6. $\frac{\text{area}}{\text{period}} = \frac{\pi ab}{T} = \frac{dA}{dt} = \frac{L}{2m}$

$$\therefore T = \frac{2m\pi ab}{L} = 2\pi mab\frac{1}{mb}\left(\frac{a}{GM}\right)^{1/2} = \frac{2\pi}{\sqrt{GM}}a^{3/2}$$

(from (5))
