M. Coppins
14.12.07

Supplementary Problem Sheet

Newton's demonstration that the motion of the planets is a direct consequence of the nature of the gravitational force was one of the turning points in the history of science. Anyone studying classical mechanics can reasonably expect to encounter a proof that Kepler's laws do indeed follow from assuming an attractive, central gravitational force proportional to r^{-2}. The questions on this Supplementary Problem Sheet lead you through the derivation.

You should regard this problem sheet as an optional extra. The questions involve equations for ellipses which lie outside the regular course. I would not set a question like this on the exam.

We do not follow Newton's method here. Instead we assume the planet follows an elliptical orbit and show that this assumption is consistent with the conservation of angular momentum and energy, which were basic properties of the gravitational force. We also find equations for these conserved quantities in terms of the characteristic parameters of the orbit, and then use this information to derive Kepler's third law.

1. The equation of an ellipse (with one focus at the origin) in polar coordinates can be written

$$r = \frac{\alpha}{(1 - e\cos\theta)}$$

where $\alpha = b^2/a$, a and b are the semi-major and semi-minor axes respectively, and e is the eccentricity (Handout 4, Eq. 7). Show that the radial component of the velocity of a particle following an elliptical orbit (i.e., a planet), with the origin at one focus, is

$$v_r = -\frac{\alpha e \sin\theta}{(1 - e\cos\theta)^2}\frac{d\theta}{dt}$$

2. We now make use of the fact that L = angular momentum = constant in a gravitational orbit. Show that

$$\frac{d\theta}{dt} = \frac{L(1 - e\cos\theta)^2}{m\alpha^2} \ ,$$

and, hence, that

$$v_r = -\frac{Le\sin\theta}{m\alpha^2} \ .$$

3. Use the polar equation of the ellipse, given in Q. 1, to show that at any point on the ellipse: $e^2\sin^2\theta = e^2 - 1 + \frac{2\alpha}{r} - \frac{\alpha^2}{r^2}$. [Hint: remember that $\sin^2\theta = 1 - \cos^2\theta$.]

Continued overleaf

4. Using the equation $e^2 = 1 - b^2/a^2$ (Handout 4, Eq. 5), the equation found in the previous part, and the equation for v_r found in Q. 2, show that

$$v_r^2 = \frac{L^2}{m^2}\left(-\frac{b^2}{\alpha^2 a^2} + \frac{2}{\alpha r} - \frac{1}{r^2}\right)$$

and, hence that the total kinetic energy can be written

$$K = -\frac{L^2}{2mb^2} + \frac{L^2 a}{mb^2 r} \ .$$

[Hint: use the definition of α given in Q. 1.]

5. We now use the fact that the total energy of the particle is constant, and that its potential energy is $U = -\frac{GMm}{r}$, where M is the mass of the object about which the particle is orbitting (i.e., the Sun). Thus the kinetic energy of the particle must have the form

$$K = E - U = E + \frac{GMm}{r}$$

where $E =$ constant. The expression for K obtained in the previous part has the correct functional form (a constant + a term proportional to r^{-1}). This indicates that our assumption of an elliptical orbit was consistent with Newton's law of universal gravitation (from which we obtained conservation of angular momentum and the equation for U), thus confirming Kepler's first law. Show that

$$L = mb\sqrt{\frac{GM}{a}} \quad \text{and} \quad |E| = \frac{GMm}{2a} \ .$$

6. In Lecture 5.2 we showed that the rate at which area is swept out by a planet is $\frac{dA}{dt} = \frac{L}{2m} =$ constant (this is Kepler's second law). Given that the area of an ellipse is πab (Handout 4, Eq. 8), show that the period of the orbit is $T = 2\pi mab/L$, and, using the equation for L obtained in the previous part, show that

$$T = \frac{2\pi}{\sqrt{GM}} a^{3/2}$$

i.e., the period of the orbit is proportional to $a^{3/2}$, where the constant of proportionality is the same for all planets (it depends only on $M =$ mass of the Sun). This is Kepler's third law.