UNIVERSITY OF LONDON

BSc/MSci EXAMINATION April 2004

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

MECHANICS & RELATIVITY

For First-Year Physics Students

Wednesday 28th April 2004: 10.00 to 12.00

Answer ALL parts of Section A and TWO questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) Quote *Newton's First Law of Motion*. Define an *inertial reference frame*. Briefly explain how, an astronaut inside a sealed space-station with no windows, in Earth orbit, could establish in principle that he/she is NOT in a perfect inertial frame.

[4 marks]

(ii) A projectile is thrown in the air at speed u at an angle α with the horizontal. Assume that g is constant and that the effects of air resistance can be neglected. Show that the *range* of the projectile (i.e. the distance it reaches in the horizontal plane) is given by

$$R = \frac{u^2 \sin 2\alpha}{g}.$$

What is the maximum range R_{max} and what is the corresponding value of α?

[5 marks]

(iii) A cricket ball of mass 0.2 kg is thrown horizontally at a speed 30 ms^{-1} by a fielder. If the ball was initially at rest, what is the work done by the fielder in throwing the ball? If the force exerted was a constant 100 N, over what distance were hand and ball in contact? Calculate the impulse given to the ball by the fielder from the rate of change of momentum. Use this result and the constant force of 100 N to estimate the time for which the ball and hand were in contact.

[4 marks]

(iv) A two-body system consists of a particle of mass m_1 at position $\underline{r}_1$ and a particle of mass m_2 at position $\underline{r}_2$ interacting by an "Action Reaction" pair of *internal forces*. In addition, particle *1* experiences an external force $\underline{F}_1^{Ext}$ and particle *2* experiences an external force $\underline{F}_2^{Ext}$. The vector $\underline{R}$ which determines the position of the *centre of mass* of the two particles is defined by:

$$\underline{R} = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1 + m_2}.$$

Show that if $M = m_1 + m_2$ the motion of the centre of mass is governed by the equation

$$M\ddot{\underline{R}} = \underline{F}_{Tot}^{Ext} = \underline{F}_1^{Ext} + \underline{F}_2^{Ext}.$$

[5 marks]

(v) A crown bowling ball of mass 0.5 kg is moving across the surface of a green with constant speed 4 ms^{-1} in the +x direction. It hits the jack, which is a smaller ball of mass 0.1 kg, and initially at rest. The original bowling ball is observed to have speed 3 ms^{-1} after the collision. Assuming the collision is elastic, what is the speed of recoil of the jack? If the ball of mass 0.5 kg leaves the collision at an angle of 10.5° with the x-direction, what angle does the recoil motion of the jack make with the x-axis? You may assume that friction can be ignored and that the ball is spherically symmetrical.

[5 marks]

(vi) Define the *torque* of a force $\underline{F}$ on a particle of mass m, which is at position $\underline{r}$ with respect to a fixed point O. Define the *angular momentum* of the particle about O. Derive the vector relationship between the torque and angular momentum.

[4 marks]

[TOTAL 27 marks]

2. A particle with rest mass m moves with speed u. The total energy, E, and momentum, p, of the particle may be written

$$E = \gamma mc^2\,; \quad p = \gamma\beta mc$$

where c is the speed of light, $\beta = u/c$ and $\gamma = \left(1 - \beta^2\right)^{-\frac{1}{2}}$.

(i) Show that, in the low velocity limit $\beta << 1$, the energy and momentum may be approximated by

$$E \approx mc^2 + \tfrac{1}{2}mu^2\,; \quad p \approx mu\,.$$

[5 marks]

(ii) An electron, rest mass $m = 500\,\mathrm{keV}/c^2$, enters a simple accelerator at a velocity of $0.01c$. An electric field is applied to accelerate the electron. The potential difference between the entrance and exit window of the device is +500 kV. Hence, the electron gains an energy of 500 keV as it passes through the accelerator.

(a) What is the total energy of the electron as it enters the accelerator? [1 mark]

(b) What is the total energy of the electron when it leaves the accelerator? [1 mark]

(c) At what velocity does the electron leave the accelerator? [2 marks]

[TOTAL 9 marks]

SECTION B

3. A student standing at the top of a tall cliff throws a ball of mass m vertically upwards with initial magnitude of velocity u so that on its return the ball just misses the cliff edge and drops down into the sea. Assume that the trajectory is up or down the vertical y-axis and that air resistance can be approximated by a force $b\upsilon_y$ where b is a positive constant and υ_y is the y-component of velocity. You may also assume that the acceleration due to gravity g is constant.

(i) Set up Newton's Second Law for the vertical motion of the ball and hence find an expression for the *terminal velocity* of the ball in terms of m, b and g. [4 marks]

(ii) By separation of variables and integration of the equation of motion, show that the vertical component of velocity $\upsilon_y(t)$ at time t is given by:

$$\upsilon_y(t) = \left(\frac{mg}{b} + u\right) e^{-\frac{bt}{m}} - \frac{mg}{b}.$$

[9 marks]

(iii) Show that $\upsilon_y(t)$ approaches the result in (i) as $t \to \infty$.

[2 marks]

(iv) Use your result for $\upsilon_y(t)$ in part (ii) to find an expression, in terms of u, m, b and g, for the time t_{max} at which the ball reaches its maximum height above the student.
[9 marks]

(v) Use the power series expression for the exponential function:

$$e^x = 1 + x + \frac{x^2}{2!} + ...$$

to show that in the limit $m >> bt_{max}$ the expression for t_{max} approaches that expected in the situation where air resistance can be neglected.

[8 marks]

[TOTAL 32 marks]

4. (i) A rocket of mass M_i is travelling freely with constant velocity υ_i in the $+x$-direction in an inertial frame. Assume that the effects of gravity and resistive forces can be ignored. In order to accelerate to a constant velocity $\upsilon_f(\upsilon_f > \upsilon_i)$ when its mass will be $M_f(M_f < M_i)$ spent fuel is ejected in the $-x$-direction at a constant speed υ_{ex} relative to the rocket.

In time dt the velocity of the rocket in the x-direction increases from υ to $\upsilon + d\upsilon$, the mass of the rocket changes from m to $(m + dm)$ and the rocket ejects a quantity of spent fuel $-dm$ (where $dm < 0$). Use the Conservation of Linear Momentum to derive the rocket equation

$$\upsilon_f - \upsilon_i = \upsilon_{ex} \ln\left(\frac{M_i}{M_f}\right)$$

explaining carefully any assumptions you make. [10 marks]

(ii) The initial fuel load of the rocket was 80% of the initial mass M_i. If the rocket accelerates from $\upsilon_i = 1\,\mathrm{kms}^{-1}$ until it reaches $\upsilon_f = 3\,\mathrm{kms}^{-1}$ when all of its fuel has been burnt, what is the constant exhaust speed υ_{ex}? [6 marks]

(iii) The rocket, of mass M_f, continues to cruise at constant velocity υ_f in the $+x$-direction. A lunar module of mass kM_f (where $k < 1$) is released from the rocket by an *inelastic* process in which a positive quantity of mechanical energy Q is generated. Use kinetic energy conservation and momentum conservation in the x-direction to show that *if the rocket velocity is reduced to rest* while the lunar module moves with velocity υ in the $+x$-direction, then

$$\upsilon^2 = \frac{2Q}{kM_f(1-k)}.$$

[9 marks]

(iv) Express the *velocity of the centre of mass* of the rocket plus lunar module system after the release in terms of υ_f. What important principle does this illustrate? [7 marks]

[TOTAL 32 marks]

5. (i) Explain briefly how energy conservation, angular momentum conservation and Newton's Law of Gravitation lead to an expression for the total energy of a planet of mass m in orbit about the sun mass M_S in terms of the radial component of velocity υ_r that is of the form:

$$E = \frac{1}{2} m \upsilon_r^2 + U_{eff}(r)$$

where the *effective potential* is

$$U_{eff}(r) = \frac{L^2}{2mr^2} - \frac{GmM_S}{r} \ .$$

[7 marks]

(ii) Sketch the effective potential and the two terms in the expression for $U_{eff}(r)$. Which term corresponds to an attractive force and which to a repulsive force? Indicate the radial position and the energy E_0 corresponding to a *circular* planetary orbit.

[7 marks]

(iii) Determine the position r_0 of the minimum of the effective potential and show that

$$r_0 = \frac{L^2}{Gm^2 M_S} \ .$$

[5 marks]

(iv) Determine the second derivative of the effective potential and show that its value at r_0 is given by

$$\left(\frac{d^2 U_{eff}(r)}{dr^2} \right)_{r=r_0} = \frac{G^4 m^7 M_S^4}{L^6} \ .$$

[9 marks]

(v) Use the result in part (iv) and the parabolic approximation for the potential function to show that the angular frequency ω of *small amplitude radial oscillations* about the minimum position r_0 is given by:

$$\omega = \frac{G^2 m^3 M_S^2}{L^3} \ .$$

[4 marks]

[TOTAL 32 marks]

6. The Lorentz transformation of space and time between two inertial frames S and S' may be written

$$x = \gamma\left(x' + \beta ct'\right), \quad y = y', \quad z = z', \quad ct = \gamma\left(ct' + \beta x'\right)$$

where the velocity with which S' moves relative to S is u in the $+x$-direction, $\beta = u/c$ (c is the speed of light) and

$$\gamma = \left(1 - \beta^2\right)^{-\frac{1}{2}}.$$

At $t = t' = 0$ the coordinate axes of S and S' coincide.

(i) A body moves at a velocity $\underline{v}'$ in S'. The components of $\underline{v}'$ along the coordinate axes in S' are v'_x, v'_y and v'_z. The body is observed to move at a velocity of $\underline{v}$ in S.

(a) Show that the components of $\underline{v}$ along the coordinate axes of S are given by

$$\frac{v_x}{c} = \frac{\Delta x}{c\Delta t} = \frac{\frac{v'_x}{c} + \beta}{1 + \beta\frac{v'_x}{c}}$$

$$\frac{v_y}{c} = \frac{\Delta y}{c\Delta t} = \frac{1}{\gamma}\frac{\frac{v'_y}{c}}{1 + \beta\frac{v'_x}{c}}$$

$$\frac{v_z}{c} = \frac{\Delta z}{c\Delta t} = \frac{1}{\gamma}\frac{\frac{v'_z}{c}}{1 + \beta\frac{v'_x}{c}}$$

where Δx, Δy, Δz and $c\Delta t$ are the infinitesimal distances between adjacent points on the worldline of the body viewed from S. [13 marks]

(b) Under what conditions do the expressions derived in part (i)(a) reduce to the Galilean transformation of velocity:

$$v_x = v'_x + u;$$
$$v_y = v'_y;$$
$$v_z = v'_z?$$

[7 marks]

(ii) In a particular radioactive decay an electron travelling at $0.8c$ relative to the nucleus is produced. The nucleus itself is travelling at $0.5c$ with respect to the laboratory in which the decay is observed. What is the velocity of the electron in the laboratory frame if:

(a) the electron is emitted parallel to the direction of motion of the nucleus; [4 marks]

(b) the electron is emitted perpendicular to the direction of motion of the nucleus; [4 marks]

(c) the electron is emitted anti-parallel to the direction of motion of the nucleus. [4 marks]

In each case, give all three components of the velocity of the electron relative to the laboratory frame.

[TOTAL 32 marks]

End