PROBLEM SHEET 9

1 (a) $7!/4! = 210$, (b) $4!/0! = 4! = 24$, (c) $7!/6! = 7$
(d) Last digit must be 1, 3 or 5. First 2 digits chosen from remaining 4 $\to 3 \times {}^4P_2 = 36$

2 (a) $7!/(3!4!) = 35$, (b) $4!/(4!0!) = 1$, (c) $7!/(6!1!) = 7$
(d) ${}^{12}C_5 = 12!/(5!7!) = 792$

3 (a) ${}^{20}C_5 = \dfrac{20!}{5!5!} = 15504$

(b) Coeff of y^3 in $(1+y)^5$ is ${}^5C_3 = 5!/3!2! = 10$
∴ term in x^3 is $10 \times (2x)^3 \to$ coeff of x^3 is 80

(c) $(3+x)^{25} = 3^{25}(1+y)^{25}$ $[y = x/3]$ Coeff of y^{22} is $3^{25} \times {}^{25}C_{22}$
But $y^{22} = x^{22}/3^{22}$ ∴ coeff of x^{22} is $\dfrac{1}{3^{22}} \times 3^{25} \times \dfrac{25!}{22!3!} = 62100$

(d) $(2-18x)^{10} = 2^{10}(1+y)^{10}$ $[y = -18x/2 = -9x]$
Coeff of y is $2^{10} \times 10$ ∴ coeff of x is $2^{10} \times 10 \times (-9) = -92160$

(e) $(1+4x)^{17} = 1 + 4.17x + \dots = 1 + 68x + \dots$
$(1-3x)^{41} = 1 - 3.41x + \dots = 1 - 123x + \dots$
$(1+68x+\dots)(1-123x+\dots) = 1 - 123x + 68x - 68.123x^2 + \dots$
all remaining terms must be x^2 or higher order
∴ coeff of x is $-123+68 = -55$

4 (a) $(1+x)^{-1} = 1 + (-1)x + \dfrac{(-1)(-2)x^2}{2!} + \dfrac{(-1)(-2)(-3)x^3}{3!} + \dfrac{(-1)(-2)(-3)(-4)x^4}{4!} + \dots$
$= 1 - x + x^2 - x^3 + x^4 - \dots$

(b) $(1-x)^{-1} = (1+y)^{-1}$ with $y = -x$.
$= 1 - y + y^2 - y^3 + y^4 - \dots = 1 + x + x^2 + x^3 + x^4 + \dots$

(c) The expansion for $(1-x)^{-1}$ was obtained by summing a G.P.

5 (a) $f(x) = \sin x$, $f(0) = 0$, $f'(x) = \cos x$ $f'(0) = 1$, $f''(x) = -\sin x$, $f''(0) = 0$
$f'''(x) = -\cos x$, $f'''(x) = -1$ $f''''(x) = \sin x$, $f''''(x) = 0$,
$f'''''(x) = \cos x$, $f'''''(0) = 1$ ∴ $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots$

(b) Small angles $\sin x \simeq x$, ok if $x^3/3! \ll x \to |x| \ll \sqrt{6} \simeq 2.4$ rad

(c) $10° = 0.17$ rad $\ll 2.4$ rad so probably ok (actually small angle formula gives value accurate to 2 sig fig's).
REMEMBER - SMALL ANGLE APPROX'S FORMULAE FOR DERIVATIVES & MACLAURIN SERIES FOR SIN & COS
ONLY VALID IF ANGLES ARE IN RADIANS

6 (a) $f(x) = \ln x$, $f(0) = -\infty$; $f'(x) = 1/x$, $f'(0) = +\infty$; $f''(x) = -1/x^2$, $f''(0) = -\infty$
$\to$ it's all gone pear-shaped.

(b) $f(x) = \ln(1+x)$, $f(0) = 0$; $f'(x) = (1+x)^{-1}$, $f'(0) = 1$;
$f''(x) = -(1+x)^{-2}$, $f''(0) = -1$; $f'''(x) = 2(1+x)^{-3}$, $f'''(0) = 2$
∴ $\ln(1+x) = x - \dfrac{x^2}{2!} + \dfrac{2x^3}{3!} - \dots = x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$

$\ln(1.05) = \ln(1+0.05) \simeq 0.05 - \dfrac{(0.05)^2}{2} + \dfrac{(0.05)^2}{3} \simeq 0.0488$

(c) In PS7 we used $\ln(1+x) \simeq x$, equivalent to only 1st term of M. series.

7 (a) $|x| < 1$, $(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{1}{2}\left(-\frac{1}{2}\right)\dfrac{x^2}{2!} + \dots = 1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \dots$

(b) $|-2x| < 1 \to |x| < 1/2$, $(1+y)^{1/10} = \dfrac{1}{10}y + \dfrac{1}{10}\left(-\dfrac{9}{10}\right)\dfrac{y^2}{2!} + \dots$

$y = -2x \Rightarrow (1-2x)^{1/10} = 1 - x/5 - \dfrac{9x^2}{50} + \dots$

(c) write as $2^{-1}(1+y)^{-1}$, $y = 3x/2 \to$ ∴ need $|y| < 1 \to |x| < 2/3$
$2^{-1}(1-y)^{-1} = \dfrac{1}{2}\left\{1 + (-1)y + \dfrac{(-1)(-2)}{2!}y^2 \dots\right\} \to \dfrac{1}{2} - \dfrac{3}{4}x + \dfrac{9}{8}x^2 + \dots$

(d) write as $a^{-2}(1+y)^{-2}$, $y = x/a$, need $|y| < 1 \to |x| < a$.
$a^{-2}(1+y^2)^{-2} = a^{-2}\left\{1 + (-2)y + \dfrac{(-2)(-3)}{2!}y^2 + \dots\right\} = \dfrac{1}{a^2} - \dfrac{2x}{a^3} + \dfrac{3x^2}{a^4} + \dots$

(e) Write as $(1+y)^{1/3}$, $y = 2x^2$, need $|y| < 1 \to |x| < 1/\sqrt{2}$
$(1+y)^{1/3} = 1 + \dfrac{1}{3}y + \dfrac{1}{3}\left(\dfrac{-2}{3}\right)\dfrac{y^2}{2!} + \dots = 1 + \dfrac{2}{3}x^2 - \dfrac{4}{9}x^4 + \dots$

8 (a) FALSE (it has no real roots) (b) TRUE
(c) FALSE ($\cos^{-1}(0) = \pi/2 = \sin^{-1}(1)$, $\sin^{-1}(\pi/2)$ is not defined)
(d) TRUE ($y'' = 4e^{2x} + 4xe^{2x}$) (e) FALSE (it is $-n\beta/\alpha^{n+1}$)