PROBLEM SHEET 7

1(a) $y+\delta y = 3(x+\delta x)^2+(x+\delta x) = 3x^2+6x\delta x+\delta^2 x + x + \delta x$ (y)

$\therefore \delta y = 6x\delta x + \delta x^2 + \delta x$.

$\therefore \frac{\delta y}{\delta x} = 6x + \delta x + 1$, $\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = 6x+1$

(b) $y+\delta y = (x+\delta x)^3 = x^3 + 3x^2\delta x + 3x\delta^2 x + \delta^3 x$.

$\therefore \frac{\delta y}{\delta x} = 3x^2 + 3x\delta x + \delta^2 x$, $\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = 3x^2$.

(c) $\delta y = \frac{1}{(x+\delta x)} - \frac{1}{x} = \frac{x-(x+\delta x)}{(x+\delta x)x} = -\frac{\delta x}{(x+\delta x)x}$ $\therefore \frac{\delta y}{\delta x} = \frac{-1}{(x+\delta x)x}$

$\therefore \frac{dy}{dx} = \frac{-1}{x\cdot x} = -\frac{1}{x^2}$

2(a) $y+\delta y = \sin(x+\delta x) = \sin x\cos\delta x + \cos x\sin\delta x$.

small $\delta x \to y+\delta y \simeq \sin x(1-\delta^2 x/2) + \cos x(\delta x)$.

$\therefore \delta y \simeq -\frac{\delta^2 x}{2}\sin x + \delta x\cos x$ (since $y=\sin x$)

$\therefore \frac{\delta y}{\delta x} \simeq \cos x - \frac{\delta x}{2}\sin x$, take $\lim \delta x\to 0$ & give $\frac{dy}{dx} = \cos x$.

(b) $y+\delta y = \cos(x+\delta x) \simeq \cos x(1-\delta^2 x/2) - \sin x(\delta x)$.

$\therefore \frac{\delta y}{\delta x} \simeq -\sin x - \frac{\delta x}{2}\cos x$, $\delta x\to 0$ gives $\frac{dy}{dx} = -\sin x$

3(a) $y=u^2$, $u=3x+2$, $\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = 2u\cdot 3 = 6(3x+2)$

(b) $y=u^{10}$, $u=7x^2+1$, $dy/dx = 10u^9\cdot 14x = 140x(7x^2+1)^9$

(c) $y=u^{-4}$, $u=2x-3$, $dy/dx = -4u^{-5}\cdot 2 = -8(2x-3)^{-5}$

(d) $y=u^n$, $u=\alpha x+\beta$, $dy/dx = nu^{n-1}\cdot\alpha = n\alpha(\alpha x+\beta)^{n-1}$

(e) $y=\ln u$, $u=\alpha x+\beta$, $dy/dx = \frac{1}{u}\cdot\alpha = \frac{\alpha}{(\alpha x+\beta)}$

4(a) $y+\delta y = (u+\delta u)(v+\delta v) = uv + u\delta v + v\delta u + \delta u\delta v$

$\therefore \delta y = u\delta v + v\delta u + \delta u\delta v$ (since $y=uv$)

$\therefore \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u\frac{\delta v}{\delta x}$. In limit $\delta x\to 0$ then $\frac{\delta u}{\delta x}\to\frac{du}{dx}$, $\frac{\delta v}{\delta x}\to\frac{dv}{dx}$ but $\delta u\to 0$ $\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

(b) $\frac{dw}{dx} = \frac{dw}{dv}\times\frac{dv}{dx} = -v^{-2}\frac{dv}{dx}$

$\frac{dy}{dx} = \frac{d}{dx}(uw) = u\frac{dw}{dx} + w\frac{du}{dx} = -\frac{u}{v^2}\frac{dv}{dx} + \frac{1}{v}\frac{du}{dx}$.

5(a) $= 2x+1$, (b) $= (4x+3)^2 + x\{2\cdot 4\cdot(4x+3)\} = (4x+3)(12x+3)$

(c) $= 21x^2(2x+1)^{-1} + 7x^3\{(-1)\cdot 2\cdot(2x+1)^{-2}\} = 7x^2(4x+3)(2x+1)^{-2}$

(d) $= 4e^{2x} + 4x\cdot 2e^{2x} = 4e^{2x}(1+2x)$

(e) $= 2\cdot 4\cdot(4x-2)\ln x + (4x-2)^2/x = (4x-2)\{8\ln x + (4x+2)/x\}$

6(a) $f' = 14x^6$ $f'' = 84x^5$ (b) $f' = 6x^{-3}$ $f'' = -18x^{-4}$

(c) $f' = 10(x+5)^9$ $f'' = 90(x+5)^8$

(d) $f' = 15e^{5x}$ $f'' = 75e^{5x}$ (e) $f' = (x+1)^{-1}$ $f'' = -(x+1)^{-2}$

7(a) $\frac{dy}{dx} = 2x-2$, $\frac{d^2y}{dx^2} = 2$ S.P. when $2x-2=0\to x=1\to y=4$

$d^2y/dx^2 > 2 \Rightarrow$ min.

(b) $\frac{dy}{dx} = 3-3x^2$, $\frac{d^2y}{dx^2} = -6x$, SP's when $x^2=1\to x=\pm 1$.

$x=-1\to y=-2$, $d^2y/dx^2 = 6\to$ min, $x=1\Rightarrow y=2$, $d^2y/dx^2 = -6\to$ max

(c) $\frac{dy}{dx} = -4x^{-2}+1$, $\frac{d^2y}{dx^2} = 8x^{-3}$ SP's when $4x^{-2}=1\to x=\pm 2$.

$x=-2$, $y=-4$, $d^2y/dx^2 = -1\to$ max, $x=2$, $y=4$, $d^2y/dx^2 = 1\to$ min

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8(a) $y=\ln x$, $dy/dx = 1/x$, $\delta y \simeq \delta x/x$ $\therefore y+\delta y \simeq \ln x + \delta x/x$.

(b) $\ln(1{\cdot}05) \simeq \ln 1 + \frac{1}{1}\times 0{\cdot}05 = 0{\cdot}05$ (Calculator $\to 0{\cdot}04879$)

(c) $y = x^{1/2}$, $\delta y \simeq \frac{1}{2}x^{-1/2}\delta x$, choose $x=25$, $\delta x = 1$ then $y=5$

$\delta y \simeq \frac{1}{2}\cdot 25^{-1/2}\cdot 1 = 1/10$ $\therefore (26)^{1/2} = y+\delta y \simeq 5{\cdot}1$ (Calculator $\to 5{\cdot}099$)