Foundation Maths for First Year Physics

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Study Guide and Problem Sheet/Classwork
Lecture 7: Differentiation

Learning Outcomes

Jargon

Increment, derivative with respect to x, differentiate, stationary points, maximum, minimum, point of inflexion.

Notation

δx, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, $f'(x)$, $f''(x)$

Concepts

How to find the gradient at a point on a curve geometrically; differentiation from first principles; the product rule for differentiating; the chain rule; how to differentiate exponentials and logarithms (to base e); how to determine the stationary points of a function; how to distinguish the different types of stationary points; how to use differentiation to find small increments.

Problems

1. Differentiate from first principles:

   (a) $y = 3x^2 + x$ (b) $y = x^3$ (c) $y = \dfrac{1}{x}$

2. (a) Consider the function $y = \sin x$. For a small increment in x there is a corresponding small increment in y. Use the following identity (proved in Problem Sheet 4): $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ and the small angle approximations (also proved in Problem Sheet 4): $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1 - \theta^2/2$, to show that $\delta y = \delta x \cos x - \dfrac{\delta^2 x}{2}\sin x$. Hence, show that $\dfrac{dy}{dx} = \cos x$.

   (b) Now consider $y = \cos x$. Using the identity (again proved in Problem Sheet 4): $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$, and the small angle approximations, find an expression for δy in terms of $\sin x$, $\cos x$ and δx, and, hence, show that $\dfrac{dy}{dx} = -\sin x$.

3. Use the chain rule to differentiate the following:

   (a) $y = (3x + 2)^2$ (b) $y = (7x^2 + 1)^{10}$ (c) $y = (2x - 3)^{-4}$

   (d) $y = (\alpha x + \beta)^n$ (e) $y = \ln(\alpha x + \beta)$

4. (a) Consider a function formed from the product of two functions, $y = uv$, where $u = f(x)$ and $v = g(x)$. If x is increased by δx there are corresponding small

increments δu, δv and δy in u, v and y. Show that $\delta y = u\delta v + v\delta u + \delta u\delta v$, and, hence, obtain the product rule for differentiating.

(b) Use the chain rule to show that, if $w = v^{-1}$ then $\frac{dw}{dx} = -v^{-2}\frac{dv}{dx}$. Use this result and the product rule to show that if $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx}$.

5. Use the product rule to differentiate the following:

   (a) $y = x(x+1)$ (b) $y = x(4x+3)^2$ (c) $y = 7x^3(2x+1)^{-1}$

   (d) $y = 4xe^{2x}$ (e) $y = (4x-2)^2 \ln x$

6. Find the first and second derivatives of the following:

   (a) $f(x) = 2x^7$ (b) $f(x) = -3x^{-2}$ (c) $f(x) = (x+5)^{10}$

   (d) $f(x) = 3e^{5x}$ (e) $f(x) = \ln(x+1)$

7. Find the first and second derivatives of the following functions. Hence, find the stationary points of each one, and determine if they are maxima, minima, or points of inflexion.

   (a) $y = x^2 - 2x + 5$ (b) $y = 3x - x^3$ (c) $y = \frac{4}{x} + x$

8. If $y = f(x)$ then a small increment in x produces a corresponding small increment in y. But for small increments $\frac{\delta y}{\delta x} \simeq \frac{dy}{dx}$, i.e., $\delta y \simeq \delta x\frac{dy}{dx}$. We can use this to find approximate values of functions.

   (a) For $y = \ln x$, show that $y + \delta y \simeq \ln x + \delta x/x$

   (b) To find an approximate value of $\ln(1.05)$ take $x = 1$ and $\delta x = 0.05$. Find y and the approximate value of δy, and, hence, estimate $\ln(1.05)$. Compare your answer with the calculator value.

   (c) Estimate $\sqrt{26}$, and, having done so, compare your answer with the calculator value.