Foundation Maths for First Year Physics

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Study Guide and Problem Sheet/Classwork
Lecture 5: Functions II

Learning Outcomes

Jargon

Roots of equations, repeated root, function of a function, inverse function, principal values, asymptote.

Notation

$f^{-1}(x)$, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$

Concepts

Graphs of functions $f(x) = a^x$ and $f(x) = \log_a x$; determining if a quadratic equation has two distinct real roots, a repeated root, or no real roots; finding the roots of a quadratic equation; writing the function of a function as a single function; breaking down a complicated function into a function of a function; the condition for a function to have an inverse; finding inverse functions; how the graphs of a function and its inverse are related; principal values of sin, cos and tan; graphs of $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

Problems

1. Show that the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is $-b/a$. (Use this result to check your answers to Q. 2.)

2. Determine if the following equations have real roots, and, if so, find them:

   (a) $2x^2 - 10x + 12 = 0$ (b) $6x^2 + x = 1$ (c) $x^2 + 8x + 16 = 0$

   (d) $5x^2 = 2x$ (e) $x^2 = 25$

3. Given $f(x) = 2x$, $g(x) = 1 + 3x$ and $h(x) = 1/x$, find the following as functions of x:

   (a) $f(g(x))$ (b) $g(h(x))$ (c) $h(g(x))$ (d) $f(g(h(x)))$ (e) $h(g(f(x)))$

4. Express each the following functions in the form $y = f(u)$ and $u = g(x)$ (i.e., break it up into a function of a function):

   (a) $y = (x + 2)^4$ (b) $y = \dfrac{1}{(3x + 1)^2}$ (c) $y = 2^{x-1}$ (d) $y = \sin(x^2)$

   (e) $y = 1 - 2\log_{10} x$

5. For each of the following functions sketch its graph and determine if it has an inverse. If so, find it.

   (a) $y = 2x + 1$ (b) $y = 2^x$ (c) $y = x^3$ (d) $y = x^4$ (e) $y = \dfrac{1}{x + 1}$

6. The principal values of $\cos^{-1} x$, i.e., the angles for which $\cos^{-1}$ is defined, are $0 \leq \cos^{-1} x \leq \pi$. The principal values of $\tan^{-1} x$ are $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$. Without using a calculator:

   (a) find $\cos^{-1}(0)$, $\cos^{-1}(2)$, $\tan^{-1}(0)$, $\tan^{-1}(1)$. [Hint: one of these is undefined.]

   (b) sketch $y = \cos^{-1}(x)$ (a.k.a. arccos).

   (c) sketch $y = \tan^{-1}(x)$ (a.k.a. arctan).

7. If the graph of a function approaches a straight line, that line is called an *asymptote* of the function. For instance, $y = 1/x$ approaches the x axis for very large values of x. The x axis is therefore an asymptote of $y = 1/x$ for $x \to +\infty$. Find the following asymptotes:

   (a) $y = \dfrac{1}{x^2}$ as $x \to +\infty$ (b) $y = \dfrac{1}{x^2}$ as $x \to 0$ (c) $y = 2^x$ as $x \to -\infty$

   (d) $y = x + \dfrac{1}{x}$ as $x \to +\infty$ (e) $y = \tan^{-1} x$ as $x \to +\infty$ (see Q. 6c).

8. Decide if the following statements are true or false:

   (a) The roots of the equations $3x - 6 = 0$ and $10 - 5x = 0$ are the same.

   (b) If $f(x) = \dfrac{1}{x}$ then $f(f(x)) = \dfrac{1}{x^2}$.

   (c) The inverse function of $\log_2 x$ is 2^x.

   (d) $\sin^{-1}(x)$ is not defined for $x > 1$.

   (e) $\tan^{-1}(x)$ is an even function.