Lecture 6

e

A simple model of population growth suggests that

Rate of increase of population $\propto$ population

(assuming plentiful food supply, no predators, etc)

Suppose we know the population at some time & the rate of increase at that time

> Can we predict the population at some time in the future?

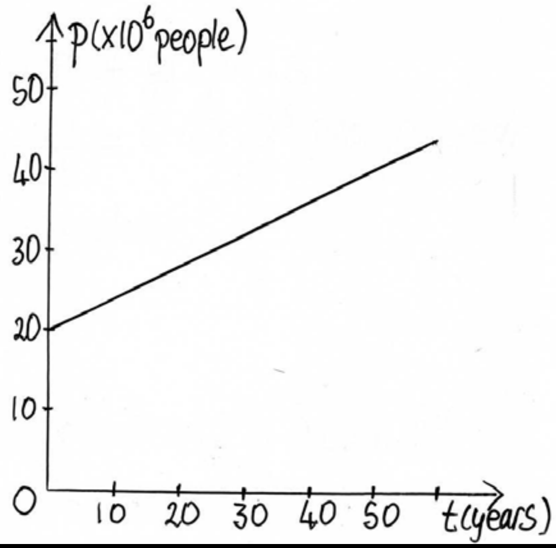
Consider a planet which has been colonized and has a current population of 20 million people, with a current rate of increase of 400,000 people/year

What is the population after t years?

FIRST ATTEMPT

Assume the rate of increase stays constant

$\rightarrow$ graph of p = population, against t would be a straight line.



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In any interval Δt years

Δp = increase in population

$= 0.4 \times 10^6 \Delta t$

e.g. $t=0$ to $t=5$, $\Delta t = 5$ years

$\rightarrow \Delta p = 2.0 \times 10^6$ people

$t=20$ to $t=40$, $\Delta t = 20$ years

$\rightarrow \Delta p = 8.0 \times 10^6$ people

Slope of graph is

$\frac{\Delta p}{\Delta t} = 0.4 \times 10^6$ people/year

everywhere.

Slope is independent of

- when interval started
- how long interval lasted.

$\rightarrow p(t) = 20 \times 10^6 + 0.4 \times 10^6 t$

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But the assumption that the rate of increase stays constant is WRONG

Rate of increase of $p \propto p$

$\boxed{\text{As } p \text{ increases, the rate of increase increases.}}$

Graph of p against t is not a straight line.

What would it look like?

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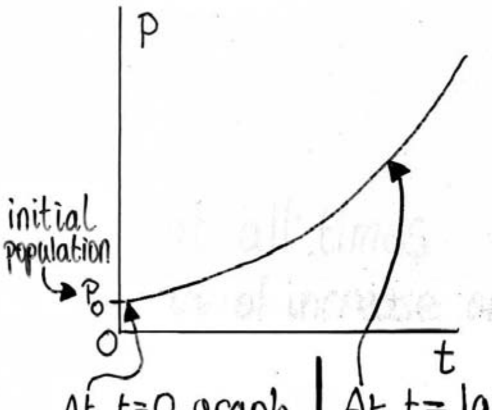

At $t=0$ graph is sloping upwards (p is increasing)

At $t=$ later graph is sloping upwards more steeply (p is growing faster because p has increased)

The graph gets steadily steeper.

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Initially, the rate of increase is 0.4×10^6 people/year.

Later the rate of increase is larger

But at all times

R = rate of increase of p

$= \gamma p$ (i.e. $\propto p$)

γ: constant

R has unit : people/year

$\therefore \gamma$ has unit years^{-1}

To find value, use data at $t=0$

$0.4 \times 10^6 = \gamma \times 20 \times 10^6$

$\rightarrow \gamma = \frac{1}{50}$ years^{-1}

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The value of γ suggests that 50 years has some special significance.

So, we will initially focus on the question: what is $p(t=50)$?

Our first attempt gave

$$p(t=50) = 20\times10^6 + 0{\cdot}4\times10^6\times50$$
$$= 40\times10^6$$

i.e. p would double.

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But this underestimates the correct value. It assumes the graph of p against t is a straight line with slope $0{\cdot}4\times10^6$ people/year.

The actual curve $p(t)$ must have this slope initially, but as t increase so p increases → the slope increases.

> The actual curve diverges from the straight line with a steadily increasing slope.

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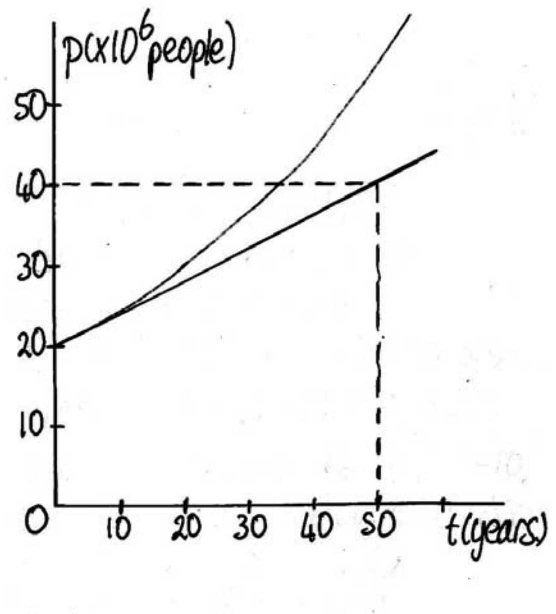

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SECOND ATTEMPT

Instead of assuming constant growth over the entire 50 years we split it up into 5 10 year intervals

- $t=0$ to $t=10$: calculate
  Δp = increment in p
  assuming growth at
  $R = R(t=0) = 0{\cdot}4\times10^6$ people/yr.
  → predicted $p(t=10) = P_0 + \Delta p$
- Adjust R using $R = \gamma p(t=10)$
- Use new R to calculate Δp for $t=10$ to $t=20$
  → predicted $p(t=20)$
- Adjust R again

⋮

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- $\Delta p_0 = \Delta p$ calculated using R at $t=0$

$$= \frac{1}{50}\, p_0\, \Delta t$$

$$= \frac{1}{50} \times 20 \times 10^6 \times 10$$

$$= 4{\cdot}0 \times 10^6$$

$$p_1 = p(t=10) = p_0 + \Delta p_0 = 24{\cdot}0 \times 10^6$$

- $\Delta p_1 = \Delta p$ calculated using R at $t=10$

$$= \frac{1}{50}\, p_1\, \Delta t$$

$$= \frac{1}{50} \times 24{\cdot}0 \times 10^6 \times 10$$

$$= 4{\cdot}8 \times 10^6 \quad (> \Delta p_0)$$

$$p_2 = p(t=20) = p_1 + \Delta p_1 = 28{\cdot}8 \times 10^6$$

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- $\Delta p_2 = \frac{1}{50}\, p_2\, \Delta t = 5{\cdot}76 \times 10^6$

$$p_3 = p(t=30) = p_2 + \Delta p_2 = 34{\cdot}56 \times 10^6$$

- $\Delta p_3 = \frac{1}{50}\, p_3\, \Delta t = 6{\cdot}912 \times 10^6$

$$p_4 = p(t=40) = p_3 + \Delta p_3 = 41{\cdot}472 \times 10^6$$

- $\Delta p_4 = \frac{1}{50}\, p_4\, \Delta t = 8{\cdot}2944 \times 10^6$

$$p_5 = p(t=50) = p_4 + \Delta p_4 = 49{\cdot}7664 \times 10^6$$

i.e. $p(t=50) = 2{\cdot}4883\, p_0 \quad (> 2p_0)$

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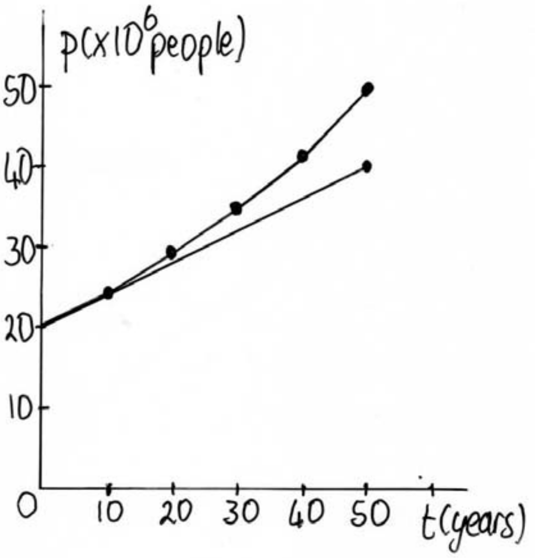

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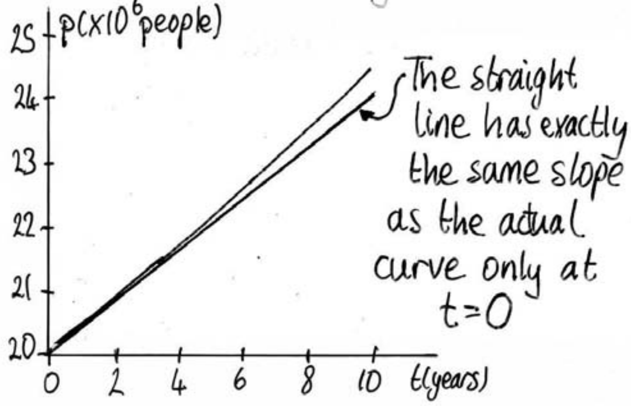
This is still inaccurate. We assumed that R stayed fixed over each 10 year interval. Actually, R increases steadily all the time.



The straight line is a tangent to the actual curve, at $t=0$

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We assumed that over the finite interval Δt (=10 years) the increment in p was:

$$\Delta p = R_{asoi}\,\Delta t$$

R at start of interval

But R is exactly R_{asoi} only a.s.o.i.

Over a SMALL interval, R doesn't change much.

For a **VERY SMALL** Δt, the increment in p is **VERY CLOSE** to $R_{asoi}\,\Delta t$.

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In the limit $\Delta t \to 0$, then $\Delta p \to 0$, but

$$\lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t} = R_{asoi}$$

[Digression: we have already seen a situation rather like this in P.S.4

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$]

Divide the 50 years into shorter intervals, the estimated value of $P(t=50)$ will be more accurate.

In the limit $\Delta t \to 0$ it will be exact.

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THIRD ATTEMPT

Use N equal steps, each of length $\Delta t = \frac{50}{N}$ years

Then let $N \to \infty$ (i.e. $\Delta t \to 0$)

- $\Delta p_0 = \frac{1}{50} p_0 \Delta t$

$$p_1 = p(t=\Delta t) = p_0 + \Delta p_0 = p_0\left(1+\frac{\Delta t}{50}\right)$$

- $\Delta p_1 = \frac{1}{50} p_1 \Delta t$

$$p_2 = p(t=2\Delta t) = p_1 + \Delta p_1 = p_1\left(1+\frac{\Delta t}{50}\right) = p_0\left(1+\frac{\Delta t}{50}\right)^2$$

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- $\Delta p_2 = \frac{1}{50} p_2 \Delta t$

$$p_3 = p(t=3\Delta t) = p_2 + \Delta p_2 = p_2\left(1+\frac{\Delta t}{50}\right) = p_0\left(1+\frac{\Delta t}{50}\right)^3$$

$$\vdots$$

$$p_n = p(t=n\Delta t) = p_0\left(1+\frac{\Delta t}{50}\right)^n$$

$$\therefore p(t=50) = p(t=N\Delta t) = p_0\left(1+\frac{\Delta t}{50}\right)^N$$

where $N = \frac{50}{\Delta t}$

i.e. $$\frac{P(t=50)}{P_0} = \left(1+\frac{1}{N}\right)^N$$

To get a really accurate result take limit $\Delta t \to 0$ i.e. $N \to \infty$

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N	Δt	$\left(1+\frac{1}{N}\right)^N$
1	50 years	2
5	10 years	2·4883
10^2	6 months	2·7048
10^4	1·83 days	2·7181
10^6	26·28 minutes	2·7183
10^8	15·77 seconds	2·7183

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$$\lim_{N\to\infty}\left(1+\frac{1}{N}\right)^N = 2{\cdot}7183\ldots = e$$

e is an irrational number

The answer to our question is:

$$p(t=50) = p_0 e = 54{\cdot}4\times10^6 \text{ people}$$

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To find p at $t=100$ we need $n=\frac{100}{\Delta t}$ steps

i.e $\frac{p(t=100)}{p_0} = \lim_{\Delta t\to 0}\left(1+\frac{\Delta t}{50}\right)^n$

still write $N=\frac{50}{\Delta t}$

then $n=2N$ and

$$\frac{p(t=100)}{p_0} = \lim_{N\to\infty}\left(1+\frac{1}{N}\right)^{2N}$$
$$= \lim_{N\to\infty}\left\{\left(1+\frac{1}{N}\right)^N\right\}^2$$
$$= e^2$$

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After 50 years the population will have increased by a factor of e.

After another 50 years it will have increased by another factor of e.

50 years is the e folding time

After m e folding times the population will have increased by a factor of e^m

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After arbitrary time

$$\frac{p(t)}{p_0} = \lim_{\Delta t \to 0} \left(1+\frac{\Delta t}{50}\right)^{n \leftarrow \frac{t}{\Delta t}}$$

$\frac{\Delta t}{50} = \frac{1}{N}$; $n = \frac{t}{\Delta t} = \frac{50}{\Delta t} \times \frac{t}{50}$

$$= \lim_{N \to \infty} \left(1+\frac{1}{N}\right)^{Nt/50}$$

$$= \lim_{N \to \infty} \left\{\left(1+\frac{1}{N}\right)^{N}\right\}^{t/50}$$

$$= e^{t/50}$$

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$$p(t) = p_0 e^{t/50}$$

Why 50?

It came from the constant in the basic eq. for the rate of increase

$$R = \gamma p \quad \gamma = \frac{1}{50} \text{ in our case}$$

In general:

$$\boxed{p(t) = p_0 e^{\gamma t}}$$

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So, how long does it take for the population to double?

i.e. what is t such that $p(t) = 2p_0$

$$\rightarrow p_0 e^{\gamma t} = 2p_0$$

$$\rightarrow \gamma t = \ln 2 \quad \leftarrow \log_e$$

$$\rightarrow t = \frac{1}{\gamma} \ln 2$$

$$= 34.66 \text{ years for } \gamma = \frac{1}{50}$$

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How long to quadruple?

$$p(t) = 4p_0$$

$$\rightarrow e^{\gamma t} = 4$$

$$\rightarrow t = \frac{1}{\gamma} \ln 4 = \frac{1}{\gamma} \ln 2^2 = \frac{2}{\gamma} \ln 2$$

$$= 69.31 \text{ years for } \gamma = \frac{1}{50}$$

Every $\frac{1}{\gamma} \ln 2$ years the population doubles.