UNIVERSITY OF LONDON [MP1 2005]

B.Sc. and M.Sci. DEGREE EXAMINATIONS 2005

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City and Guilds of London Institute

FIRST YEAR STUDENTS OF PHYSICS

MATHEMATICS - M. PHYS 1

Date: Thursday 28th April 2005 Time: 10 am - 1 pm

Do not attempt more than SIX questions

[*Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) Find the limits:

(a)
$$\lim_{x\to\frac{\pi}{2}} \tan x \left\{(\tan^2 x + 1)^{1/2} - (\tan^2 x - 1)^{1/2}\right\}$$

with the help of the substitution $y = \tan x$;

(b)
$$\lim_{x\to\pi} \frac{\tan^2 x}{1+\cos x} .$$

(ii) Sketch the graph
$$y = \frac{x(x+6)}{x-2} ,$$

determining the location of any zeros, asymptotes and stationary points.

Also determine how the function behaves for small and large values of x.

2. (i) Given the equation of a curve in polar co-ordinates,
$$r(\theta) = 1 + \cos\theta ,$$

determine the total length of the curve between $\theta = 0$ and $\theta = \pi$.

(ii) Show that
$$(\cos x \cos y + 2xy)\, dx + (x^2 - \sin x \sin y)\, dy$$

is an exact differential and determine the function $u(x, y)$, for which it is the total derivative.

Hence, or otherwise, solve the differential equation
$$\frac{dy}{dx} = \frac{\cos x \cos y + 2xy}{\sin x \sin y - x^2} .$$

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3. Consider the function of two variables

$$u(x, y) \;=\; f(x-y) \;+\; g\left(x + \frac{1}{3}y\right) \ ,$$

where $f(s)$ and $g(t)$ are each arbitrary functions of a single variable.

Using the change of variables

$$s \;=\; x - y \ ,$$

$$t \;=\; x + \frac{1}{3}\,y \ ,$$

use the chain rule to determine the first and second derivatives of u with respect to x and y in terms of derivatives of f and g.

Hence, show that the second derivatives satisfy

$$u_{xx} \;=\; 2u_{xy} \;+\; 3u_{yy} \quad \text{where} \quad u_{xx} \;=\; \partial^2 u/\partial x^2 \text{ etc.}$$

4. If $\qquad z \;=\; \dfrac{-4}{1+i\sqrt{3}}$,

(i) find the real and imaginary part of z ;

(ii) find the modulus and argument of z ;

(iii) find the modulus and argument of z^2 ;

(iv) find the moduli and arguments of all values of $z^{1/3}$;

(v) plot the results of (i) - (iv) on a rough sketch of the complex plane.

Quote arguments within the range $0 \leq \theta < 2\pi$ in both degrees and radians.

5. (i) What geometrical object does the equation $2x - 3y + z = 4$ represent ?
In what direction relative to the object is the vector $2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ directed ?

(ii) If $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$, find $\boldsymbol{a} \times \boldsymbol{b}$ and $|\,\boldsymbol{a} \times \boldsymbol{b}\,|$.

Explain the geometrical significance of $|\,\boldsymbol{a} \times \boldsymbol{b}\,|$ in relation to a geometrical object defined by $\boldsymbol{a}$ and $\boldsymbol{b}$, which you should describe.

(iii) If $\boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j} + \alpha\boldsymbol{k}$, find $(\boldsymbol{a} \times \boldsymbol{b}) \bullet \boldsymbol{c}$ in the case where $\alpha = 2$. Explain the geometrical significance of $|\,(\boldsymbol{a}\times\boldsymbol{b}) \bullet \boldsymbol{c}\,|$ in relation to a geometrical object defined by $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, which you should describe.

(iv) Find $(\boldsymbol{b} \times \boldsymbol{c}) \bullet \boldsymbol{a}$ and comment on the result.

(v) For what value of α is $(\boldsymbol{a} \times \boldsymbol{b}) \bullet \boldsymbol{c} = 0$? What can you say about the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in this case ?

6. Consider the 2×2 matrix

$$\mathbf{T} = \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix} .$$

(i) Find the transformed vectors $\mathbf{s} = \mathbf{T}\widehat{\mathbf{r}}$ for the unit vectors

$$\widehat{\mathbf{r}} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} , \quad \widehat{\mathbf{r}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \text{and} \quad \widehat{\mathbf{r}} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} .$$

Write down the magnitude of $\mathbf{s}$ in each of the three cases.

(ii) Find the eigenvalues and normalised eigenvectors of $\mathbf{T}$. Explain the significance of the eigenvalues and eigenvectors in relation to part (i) of the question.

(iii) Use the normalised eigenvectors to construct a 2×2 rotation matrix $\mathbf{V}$ such that $\mathbf{V}^{-1}\,\mathbf{T}\mathbf{V} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix containing the eigenvalues of $\mathbf{T}$.

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7. State Stokes's theorem, describing the regions over which the integrations are carried out and the quantities that are being integrated.

   Evaluate the line integral

$$\int_{\mathcal{C}} \boldsymbol{V} \bullet d\boldsymbol{r} \ ,$$

   where $\mathcal{C}$ is the triangular closed path consisting of the straight lines going from the point (0, 0) to (1, 0), the point (1, 0) to (0, 1) and the point (0, 1) back to (0, 0) and

$$\boldsymbol{V} \;=\; -\,xz\,\boldsymbol{i} \;+\; (xy \;+\; yz)\,\boldsymbol{j} \;+\; xz\,\boldsymbol{k}\ .$$

   Check this result by using Stokes' theorem using the interior of $\mathcal{C}$ and taking $\boldsymbol{k}$ as the normal direction to the surface $\mathcal{S}$ capping the path $\mathcal{C}$.

8. State the divergence theorem. Your answer should include the integrations that are carried out and the quantities that are integrated. Consider the volume V bounded below by the x-y plane, the cylinder $x^2 + y^2 = 1$, and the plane $z = 2$.

   Calculate the flux of the vector field

$$\boldsymbol{F} \;=\; x\,\boldsymbol{i} \;+\; y\,\boldsymbol{j} \;+\; z^2\,\boldsymbol{k}$$

   through the surface of V.

   Use the divergence theorem to check this result.

9. Newton's law of cooling states that the temperature of a body changes at a rate proportional to the difference in temperature between the body and its environment. Thus, the differential equation for the temperature $T(t)$ of a body at time t in an environment whose ambient temperature is θ is

$$\frac{dT}{dt} = -k(T - \theta) ,$$

where k is a positive constant.

(i) By separating the variables and integrating, show that the solution of this equation with the initial condition $T(0) = T_0$ is

$$T(t) = \theta + (T_0 - \theta)e^{-kt} .$$

(ii) An alternative method of solving this equation is to write the differential equation for $u(t) = T(t) - \theta$, using the fact that θ is a constant. Show that

$$\frac{du}{dt} = -ku$$

and that the initial condition is $u(0) = T_0 - \theta$.
Solve this equation with the initial condition and obtain the same solution as in (i).

(iii) What is the temperature of the body in the limit that $t \to \infty$?

10. Suppose that the differential equation for the trajectory of a particle in the x-y plane, $\boldsymbol{r}(t) = x(t)\,\boldsymbol{i} + y(t)\boldsymbol{j}$, is given by

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{r} \times \boldsymbol{A} \ ,$$

where the vector $\boldsymbol{A}$ is given by $\boldsymbol{A} = \omega\boldsymbol{k}$.

(i) Show that the differential equations for the x- and y-components of the trajectory are given by

$$\frac{dx}{dt} = \omega y \ , \qquad \frac{dy}{dt} = -\,\omega x \ .$$

(ii) By taking appropriate derivatives, show that these coupled equations can be reduced to two second-order equations and obtain the general solutions

$$x(t) = A\cos\omega t + B\sin\omega t, \qquad y(t) = C\cos\omega t + D\sin\omega t \ ,$$

where A, B, C and D are constants.

(iii) Show that consistency with the original differential equations requires that

$$C = B, \qquad D = -\,A \ .$$

(iv) Obtain the solutions for the initial conditions

$$x(0) = 1, \qquad y(0) = 0 \ .$$

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