UNIVERSITY OF LONDON [MP1 2007]

B.Sc. and M.Sci. DEGREE EXAMINATIONS 2007

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City and Guilds of London Institute

FIRST YEAR STUDENTS OF PHYSICS

MATHEMATICS - M. PHYS 1

Date: Thursday 26th April 2007 Time: 10 am - 1 pm

Do not attempt more than SIX questions

Please use a separate answerbook for each question.

[*Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) Find the limits:

(a)
$$\lim_{x \to \infty} x^{3/2} \left[\left(1 + x + \frac{1}{x} \right)^{1/2} - (1 + x)^{1/2} \right] ;$$

(b)
$$\lim_{x \to 2} \frac{\sin(\,(x+2)^{1/2} - 2)}{x - 2} .$$

(ii) Sketch the graph

$$y = \frac{x^2 + x - 6}{x^2 + x - 2} ,$$

determining any zeros, discontinuities, maxima and minima and the limiting behaviour as $x \to \pm\infty$.

(iii) Differentiate from <u>first principles</u>

$$y = \sqrt{1 + e^x} .$$

You may find it useful to use the identity

$$a - b = \frac{a^2 - b^2}{a + b} .$$

2. (i) Consider the definite integrals

$$I_n = \int_0^1 \sinh^n x \, dx ,$$

where n is an integer.

(a) Evaluate I_1, I_2 and I_3.

Leave your answers in terms of hyperbolic functions such as $\cosh 1$, $\sinh 2$ etc.

(b) More generally show that

$$n \, I_n + (n-1) \, I_{n-2} = \cosh 1 \, \sinh^{n-1} 1 .$$

(ii) Determine the centre of mass of a uniform quarter ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ,$$

defined in the upper righthand quadrant.

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3. (i) In two dimensions, the Laplacian is defined by

$$\nabla^2 u \;\equiv\; \frac{\partial^2 u}{\partial x^2} \;+\; \frac{\partial^2 u}{\partial y^2} \;,$$

where $(x,\, y)$ are Cartesian co-ordinates.

Show that in polar co-ordinates

$$r \;=\; \sqrt{x^2 + y^2} \;, \qquad \theta \;=\; \tan^{-1}\frac{y}{x} \;,$$

the Laplacian is

$$\nabla^2 u \;=\; \frac{\partial^2 u}{\partial r^2} \;+\; \frac{1}{r}\frac{\partial u}{\partial r} \;+\; \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \;.$$

(ii) Find the stationary points of

$$u(x,\, y) \;=\; 2x^4 \;+\; 8x^2y^2 \;-\; 4x^2 \;+\; 4y^2 \;,$$

and determine their nature.

4. Consider the complex number $z \;=\; \dfrac{18+i}{2+3i}$.

(i) Find $\mathrm{Re}\{z\}$, $\mathrm{Im}\{z\}$, $|z|$ and $\arg\{z\}$.

(ii) Find the modulus and argument of z^2 .

(iii) Find the modulus and argument of z^{-1} .

(iv) Find the moduli and arguments of all values of $z^{-1/3} \;\equiv\; \sqrt[3]{z^{-1}}$.

(v) Plot the results of parts (iii) and (iv) on a rough sketch of the complex plane.

(vi) Find the real and imaginary parts of $\ln z$.

Quote arguments in radians within the range $0 < \theta < 2\pi$.

5. Vectors $\boldsymbol{A} = 2\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{B} = 3\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{C} = -5\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ define the positions of points A, B and C with respect to the origin O.

   (i) Find the angle (in degrees) between $\boldsymbol{A}$ and $\boldsymbol{B}$.

   (ii) Find the equation of the plane that passes through O, A and B.

   (iii) Find the equation of the parallel plane that passes through C, and the perpendicular distance between the two planes.

   (iv) Find the area of the parallelogram whose sides are $\boldsymbol{A}$ and $\boldsymbol{B}$.

   (v) Find the volume of the parallelepiped whose sides are $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$.

   (vi) What can you say about a point D, defined by a vector $\boldsymbol{D}$ that satisfies the equation $|\boldsymbol{D} \cdot \boldsymbol{B} \times \boldsymbol{C}| = 0$?

6. Consider the linear transformation $\mathbf{T} = \begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix}$.

   (i) If $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, find $\mathbf{v} = \mathbf{T}\,\mathbf{u}$.

   (ii) Find $\mathbf{T^{-1}}$ and verify that $\mathbf{u} = \mathbf{T^{-1}v}$.

   (iii) Find the eigenvalues and normalised eigenvectors of $\mathbf{T}$. What is the relevance of the result of part (i)? What can you deduce about the eigenvalues and eigenvectors of $\mathbf{T^{-1}}$?

   (iv) Find the angle between the eigenvectors of $\mathbf{T}$. Draw a rough graph showing the directions of the eigenvectors.

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7. Species A decays at rate α into species B and species B decays at rate β into species C (*see diagram*).

   (i) Write down the differential equations governing the time-dependent populations $A(t)$, $B(t)$ and $C(t)$ of the three species.

   (ii) Show that, if $\alpha \neq \beta$, then

   $$A(t) \;=\; A(0)e^{-\alpha t} \qquad \text{and} \qquad B(t) \;=\; B(0)e^{-\beta t} \;+\; \alpha A(0)\left(\frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha}\right).$$

   If $B(0) \;=\; 0$ and $C(0) = 0$, show that $C(\infty) \;=\; A(0)$.

   (iii) Draw a rough graph of $B(t)$ in the case where $B(0) = 0$ and α is substantially greater than β. Interpret the main features of the curve in terms of the differential equations.

   (iv) Solve the differential equation for $B(t)$ in the special case where $\alpha = \beta$, and show that $B(t) = A(0)\,(\alpha t + \rho)e^{-\alpha t}$ where $\rho = B(0)\,/\,A(0)$.

8. Consider the vector field

$$\boldsymbol{V} \;=\; y\exp(xy)\boldsymbol{i} \;+\; x\exp(xy)\boldsymbol{j}\ .$$

(i) Show that $\boldsymbol{V}$ is conservative.

(ii) Evaluate the line integral

$$\int_{\mathcal{C}} \boldsymbol{V}\cdot \mathrm{d}\boldsymbol{r}$$

where $\mathcal{C}$ is a simple path of your own choice between the points $A = (0,\, 0)$ and $B = (1,\, 1)$.

(iii) Construct a potential $F(\boldsymbol{r}) \equiv F(x,\, y)$ for the field $V(\boldsymbol{r})$.

(iv) Verify that

$$F(1,\, 1) \;-\; F(0,\, 0) \;=\; \int_{\mathcal{C}} \boldsymbol{V}\cdot \mathrm{d}\boldsymbol{r}$$

where the line integral is calculated in part (ii).

9. State Stokes' theorem. Describe the regions over which the integrations are carried out and the quantities that are being integrated.

Evaluate the line integral

$$\oint_{\mathcal{C}} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{r}\ ,$$

where

$$\boldsymbol{V} \;=\; (x \;+\; x^2y \;+\; z^2)\boldsymbol{i} \;+\; xy^2\boldsymbol{j} \;+\; xz\boldsymbol{k}$$

and where $\mathcal{C}$ is a closed path in the plane $z = 0$, consisting of four pieces:

(a) a straight line running from the point $(1,\, 0)$ to the point $(1,\, 1)$,

(b) a straight line running from the point $(1,\, 1)$ to the point $(0,\, 1)$,

(c) a semi-circle in the second and third quadrants running from the point $(0,\, 1)$ to the point $(0,\, -1)$ and finally

(d) a straight line running from the point $(0,\, -1)$ to the point $(1,\, 0)$.

Check this result by using Stokes' theorem written for the interior of $\mathcal{C}$ and taking $\boldsymbol{k}$ as the normal direction to the surface $\mathcal{S}$ capping the path $\mathcal{C}$.

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10. State the divergence theorem. Describe the regions over which the integrations are carried out and the quantities that are being integrated.

Consider the vector field

$$\boldsymbol{V} \;=\; \frac{3y\boldsymbol{i} \;+\; 2xz\boldsymbol{j} \;-\; z\boldsymbol{k}}{\sqrt{x^2 \;+\; y^2 \;+\; z^2}} \;.$$

Find the outward flux across the boundary of the hemispherical volume bounded from below by the spherical surface $x^2 + y^2 + z^2 = R^2$ (for $z < 0$) and from above by the $x - y$ plane.

Use the divergence theorem to verify the result.

END OF PAPER