UNIVERSITY OF LONDON [MP1 2004]

B.Sc. and M.Sci. DEGREE EXAMINATIONS 2004

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City and Guilds of London Institute

FIRST YEAR STUDENTS OF PHYSICS

MATHEMATICS - M. PHYS 1

Date: Thursday 29th April 2004 Time: 10 am - 1 pm

Do not attempt more than SIX questions

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

[MP1 2004]

1. (i) Evaluate the limits:

(a)
$$\lim_{x\to 1} \frac{\sin(\ln x)}{\cos\left(\frac{\pi}{2}x\right)} \ ;$$

(b)
$$\lim_{x\to\infty} x^2 \left[(1+x^3)^{1/3} - x\right] \ .$$

(ii) Sketch the function

$$y = \frac{x(x-3)}{x-4} \ ,$$

determining the location of any zeros, asymptotes, maxima and minima. Also determine how the function behaves for small and large x.

2. (i) Find the indefinite integral

$$\int \frac{x^5 dx}{1+x^2} \ .$$

(ii) Determine the path length of the function

$$y = \frac{x^3}{3} + \frac{1}{4x}$$

between $x = 1$ and $x = 2$.

3. (i) Show that there are three stationary points of the function

$$f(x, y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1$$

and determine their nature.

(ii) Is the following an exact differential?

$$(e^y + y\,e^x)\,dx + (e^x + x\,e^y - 1)\,dy \ .$$

If so, of what function?

PLEASE TURN OVER

4. (i) The nth roots of unity are $z_1, z_2, z_3, \ldots z_n$. Write down an expression for z_k, the general term in the sequence, and show that the roots can be written as a geometric sequence of the form $1, w, w^2, \ldots w^{n-1}$. Obtain an expression for w, the ratio of successive terms.

(ii) A finite geometric series of n terms can be summed:

$$S = a + ax + ax^2 + \ldots + ax^{n-1} = \frac{a(1-x^n)}{(1-x)} . \quad \text{DO NOT PROVE}$$

Use this result to show that the sum of the nth roots of unity is zero.

(iii) Find the real and imaginary parts of the three values of $1^{1/3}$. Sketch them in the complex plane and check that the real and imaginary parts of the sum of the three values are both zero.

5. (i) If the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent, then any 3-dimensional vector can be expressed as $\boldsymbol{r} = \alpha\, \boldsymbol{u} + \beta\, \boldsymbol{v} + \gamma\, \boldsymbol{w}$.

Show that

$$\alpha = \frac{\boldsymbol{r}.(\boldsymbol{v} \times \boldsymbol{w})}{\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w})}, \quad \beta = \frac{\boldsymbol{r}.(\boldsymbol{w} \times \boldsymbol{u})}{\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w})}, \quad \gamma = \frac{\boldsymbol{r}.(\boldsymbol{u} \times \boldsymbol{v})}{\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w})} .$$

(ii) Show that the vectors $\boldsymbol{u} = \boldsymbol{i}+\boldsymbol{j}$, $\boldsymbol{v} = \boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{w} = \boldsymbol{i}+\boldsymbol{k}$ are linearly independent.

(iii) Find the values of α, β and γ required to write the vector $\boldsymbol{r} = 4\boldsymbol{i}$ in terms of the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ defined in part (ii).

6. (i) Find the eigenvalues and normalized eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} .$$

(ii) Write down the matrix $\mathbf{B} = \mathbf{A}^2\ (\equiv \mathbf{AA})$.

(iii) Find the eigenvalues and normalized eigenvectors of the matrix $\mathbf{B}$, and comment on how they are related to the eigenvalues and normalized eigenvectors of $\mathbf{A}$.

7. State Stokes's theorem, identifying the regions over which the integrations are carried out and the quantities that are being integrated.
Consider the line integral

$$\int_C [P\,dx \;+\; Q\,dy \;+\; R\,dz] \ ,$$

where P, Q and R are functions of x, y and z. Determine the criterion for this integral to be independent of the path C connecting given initial and final points in terms of the curl of an appropriate vector. How does this criterion imply the existence of a potential?

(i) Consider the vector field

$$\boldsymbol{V} \;=\; y\boldsymbol{i} \ .$$

Evaluate the line integral

$$\int_C \boldsymbol{V}\cdot d\boldsymbol{r} \ ,$$

where C is the circle of radius R in the $x-y$ plane centred at the origin, and the integral is taken in the counterclockwise direction (looking down from the positive z-axis).

(ii) Evaluate the integral

$$\int\!\!\int_S (\boldsymbol{\nabla}\times\boldsymbol{V})\cdot d\boldsymbol{S} \ ,$$

where S is the curved surface of the upper half-sphere of radius R centred at the origin.

(iii) What does Stokes's theorem say about the integrals in (i) and (ii)?
Suppose S is now the surface of a circular cylinder, open at the bottom and closed at the top, of height h, whose base is a circle of radius R in the x-y plane centred at the origin. What is the value of the surface integral in (ii) over the new surface S?

PLEASE TURN OVER

8. State the divergence theorem. Your answer should describe the integrations that are carried out and the quantities that are integrated. Consider the volume V bounded below by the x-y plane and above by the upper half-sphere $x^2 + y^2 + z^2 = 4$ and inside the cylinder $x^2 + y^2 = 1$.

   Given the vector field
   $$\boldsymbol{A} = x\boldsymbol{i} + y\boldsymbol{j} + x\boldsymbol{k} ,$$
   use the divergence theorem to calculate the flux of $\boldsymbol{A}$ out of V through the spherical cap on the cylinder.

9. Consider a system composed of two species X and Y with fractional populations x and y, respectively, where $x + y = 1$. The two species interact in such a way that the differential equation for x is
   $$\frac{dx}{dt} = A(t)xy ,$$
   where $A(t) = A_0 e^{-\alpha t}$, and α and A_0 are non-negative constants. Solve the equation by separation of variables and hence show that the solution for $x(0) = x_0$ is
   $$x(t) = \frac{x_0 \exp[A_0(1 - e^{-\alpha t})/\alpha]}{1 - x_0 + x_0 \exp[A_0(1 - e^{-\alpha t})/\alpha]} .$$
   Use this solution to obtain the following:

   (i) The value of $x(t)$ as $t \to \infty$ for $\alpha > 0$.

   (ii) The value of $x(t)$ if we first let $t \to \infty$ and then let $\alpha \to 0$.

   (iii) By using L'Hôpital's rule, or otherwise, show that for finite t, in the limit $\alpha \to 0$, the time-dependent solution becomes
   $$x(t) = \frac{x_0 \, e^{A_0 t}}{1 - x_0 + x_0 \, e^{A_0 t}} .$$

10. Solve the differential equation

$$\frac{dx}{dt} + \lambda x = Ae^{i\omega t}$$

and hence show that the solution with the initial condition $x(0) = x_0$ is

$$x(t) = (x_0 - B)e^{-\lambda t} + Be^{i\omega t} ,$$

where

$$B = \frac{A}{\lambda + i\omega} .$$

Use this solution to solve the differential equation

$$\frac{dx}{dt} + \lambda x = A\cos\omega t$$

with the initial condition $x(0) = x_0$.

END OF PAPER