Problems for Lecture 9: Answers

1. (a) Since we may expand the determinant by any row or any column, we choose to expand by the third column to take advantage of the zeros. We find

$$\begin{vmatrix} 4 & 1 & 2 \\ 7 & 2 & 0 \\ -2 & 3 & 0 \end{vmatrix} = 2\begin{vmatrix} 7 & 2 \\ -2 & 3 \end{vmatrix} = 2(21+4) = 50.$$

(b) We use property 6 for determinants and multiply the second column by (-2) and add to the third column and then expand by the third column:

$$\begin{vmatrix} 3 & 2 & 4 \\ 5 & 4 & 8 \\ 8 & 2 & 9 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 0 \\ 5 & 4 & 0 \\ 8 & 2 & 5 \end{vmatrix} = 5\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = 5(12-10) = 10.$$

(c) By expanding the determinant by the first column and similarly, the determinant of its minor by the first column and so on, we find

$$\begin{vmatrix} 2 & 15 & -37 & 8 & 11 \\ 0 & 1 & 6 & 23 & -32 \\ 0 & 0 & 4 & 12 & -29 \\ 0 & 0 & 0 & 10 & 20 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 6 & 23 & -32 \\ 0 & 4 & 12 & -29 \\ 0 & 0 & 10 & 20 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 2\cdot 1\begin{vmatrix} 4 & 12 & -29 \\ 0 & 10 & 20 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= 2\cdot 1\cdot 4\begin{vmatrix} 10 & 20 \\ 0 & 3 \end{vmatrix} = 2\cdot 1\cdot 4\cdot 10\cdot 3 = 240$$

that is, the product of the diagonal elements. Indeed, in a so-called *triangular matrix*, where the elements below or above the main diagonal are zero, the determinant is simply the product of the (main) diagonal elements.
Notice that you may apply property 6 of the determinants and use the last row to create zeros in the last column above the last row, then use the second to last row to create zeros in the second to last column above that row etc. until you finally have the determinant of a *diagonal matrix*, where the only non-zero elements are in the main diagonal. Clearly the determinant of a diagonal matrix is the product of the elements in the main diagonal.

2. (a) Column 3 is 3 times column 1, so the determinant is zero by property 5.

(b) Expanding by the first column yields $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 0 \end{vmatrix} = 2\begin{vmatrix} -1 & 0 \\ 5 & 3 \end{vmatrix} = -6$.

(c) The first row is the sum of rows 2 and 3. So subtracting rows 2 and 3 in succession from row 1 makes all elements in the top row equal to zero and hence its determinant equals zero.

(d) Expanding by the first row yields $\begin{vmatrix} 0 & 7 & 0 \\ 3 & -5 & 6 \\ 2 & 3 & -4 \end{vmatrix} = -7\begin{vmatrix} 3 & 6 \\ 2 & -4 \end{vmatrix} = -7(-12-12) = 168$.

3. (a) If you change the sign of all elements in the matrix, the sign of the determinant should be reversed because each term has three factors (an odd number). This can also be obtained by applying property 2: $\det(-\mathbf{A}) = (-1)^n \det \mathbf{A}$. For this particular matrix in question 3(a), changing the sign leads to the transpose of the original matrix, that is, $\mathbf{A}^t = -\mathbf{A}$. However, by property 7, $\det \mathbf{A} = \det \mathbf{A}^t$, yielding $\det \mathbf{A} = \det(-\mathbf{A}) = -\det \mathbf{A}$ Hence, the determinant of $\mathbf{A}$ can only be zero.

(b) Using property 8: $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix}$.

Let us investigate the second determinant. We assume that $a,b,c,d \neq 0$. Then we find

$$\begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix} = \frac{1}{abcd}\begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}.$$

where step 1 involves multiplying rows 1-4 by a,b,c,d respectively (using property 2), step 2 involves applying property 2 on column 4 and step 3 involves an odd number of column exchanges, which reverses the sign (property 1). Substituting this result into the equation above, we see that the determinant of the original matrix is zero! (There are probably other ways (may be even simpler) of obtaining this result.)

If, say $a = 0$, is the determinant zero? Let me know as I havn't done the calculation myself! ☺