Problems for Lecture 8: Answers

1. (a) We evaluate the determinant of the matrix of coefficients:

$\det \mathbf{A} = \begin{vmatrix} 8 & 1 & 8 \\ 6 & 4 & 4 \\ 5 & -1 & 6 \end{vmatrix} = 8\begin{vmatrix} 4 & 4 \\ -1 & 6 \end{vmatrix} - 1\begin{vmatrix} 6 & 4 \\ 5 & 6 \end{vmatrix} + 8\begin{vmatrix} 6 & 4 \\ 5 & -1 \end{vmatrix} = 8\cdot 28 - 16 + 8\cdot(-26) = 0.$ The determinant of the matrix of coefficients is zero and there is no unique solution. To determine whether there are no solutions or infinitely many solutions, we apply Gauss elimination: multiplying the second equation with -2 and adding to the first equation, and multiplying the second equation with $-3/2$ and adding to the third equation yields the equivalent system:

$$\begin{aligned} -4x_1 \quad\; - 7x_2 &= -4 \\ 6x_1 + 4x_2 + 4x_3 &= 8 \\ -4x_1 \quad\; - 7x_2 &= 3 \end{aligned}$$

where the first and third equations are clearly incompatible. Hence there are no solutions to this system of equations.

(b) The is no unique solution. (If you insist on applying Cramer's rule to arrive at this conclusion, add a third equation $0x_1 + 0x_2 + 0x_3 = 0$ and show that the determinant of the associated matrix of coefficients is zero.) The two equations each specify a plane in $\mathbb{R}^3$. These planes are not parallel (since their respective normal vectors $(4,7,-2)$ and $(1,-3,2)$ are not parallel) so they will intersect in a line. Applying Gauss elimination, multiplying the second equation with -4 and adding to the first equation, we find

$$\begin{aligned} 19x_2 - 10x_3 &= -14 \\ x_1 - 3x_2 + 2x_3 &= 6 \end{aligned}$$

. Now, multiplying the first equation by $3/19$ and adding to the second, we find

$$\begin{aligned} 19x_2 - 10x_3 &= -14 \\ x_1 \quad\; + \frac{8}{19}x_3 &= \frac{72}{19} \end{aligned}$$

, that is,

$$\begin{aligned} x_1 &= \frac{72}{19} - \frac{8}{19}x_3 \\ x_2 &= -\frac{14}{19} + \frac{10}{19}x_3 \end{aligned}$$

obtained by interchanging the two equations and multiplying the new second equation with $1/19$ and rearranging. Hence, given x_3, x_1 and x_2 are determined by these two equations. We may write this system of equations on vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 72/19 \\ -14/19 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -8/19 \\ 10/19 \\ 1 \end{pmatrix} \Leftrightarrow \mathbf{r} = \mathbf{r}_0 + \lambda \mathbf{d}$$

where λ is a real number, revealing that the solution is a line through the point $\mathbf{r}_0$ along the direction of $\mathbf{d}$.

(c) We evaluate the determinant of the matrix of coefficients:

$$\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -2 & 2 \\ 4 & -4 & -4 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ -4 & -4 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 4 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 4 & -4 \end{vmatrix} = 16+16+0 = 32$$. Since $\det \mathbf{A} \neq 0$, there is a unique solution. Applying Cramer's rule, we find $x_1 = \dfrac{\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ -1 & -4 & -4 \end{vmatrix}}{\det \mathbf{A}} = \dfrac{16-2-2}{32} = \dfrac{3}{8}$,

$x_2 = \dfrac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 4 & -1 & -4 \end{vmatrix}}{\det \mathbf{A}} = \dfrac{2+16-2}{32} = \dfrac{1}{2}$, and $x_3 = \dfrac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \\ 4 & -4 & -1 \end{vmatrix}}{\det \mathbf{A}} = \dfrac{2+2-0}{32} = \dfrac{1}{8}$ so the unique solution is $(x_1, x_2, x_3) = (3/8, 1/2, 1/8)$ which is easily checked by substituting into the original system of equations.

2.

(a) Substituting $y=0$ into the two equations yields $x+z=8$ and $2x+3z=7$, with the solution $x=17, z=-9$. Therefore, the coordinates of the point where the line intersects the plane $y=0$ is $(17,0,-9)$ or $\mathbf{r_0} = 17\mathbf{i} - 9\mathbf{k}$.

(b) The coefficients in front of x, y, z determine a normal vector to each plane. Hence $\mathbf{n}_1 = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ or, indeed, multiples thereof are normal vectors to plane 1 and plane two, respectively.

(c) A vector $\mathbf{d}$ directed along the line of intersection clearly lies in both planes and therefore it is perpendicular to both the normal vectors. Therefore, we may use $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$.

Taking the cross product we find $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 8\mathbf{i} - \mathbf{j} - 5\mathbf{k}$.

(d) The equation for the line of intersection is determined by a point on the line, $\mathbf{r}_0$, and a direction vector for the line, $\mathbf{d}$. Hence, the equation for the line of intersection is $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d} = 17\mathbf{i} - 9\mathbf{k} + \lambda(8\mathbf{i} - \mathbf{j} - 5\mathbf{k}) = (17+8\lambda)\mathbf{i} - \lambda\mathbf{j} - (9+5\lambda)\mathbf{k}$ or, using column vectors: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17 \\ 0 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -1 \\ -5 \end{pmatrix}$. Isolating λ from the three associated equations, we find

$\lambda = \dfrac{x-17}{8} = \dfrac{y}{-1} = \dfrac{z+9}{-5}$. Note the direction ratios given by the coordinates of $\mathbf{d}$ from part (c) are in the denominators, and the coordinates of $\mathbf{r}_0$ from part (a) in the numerators.

3. Let us expand the determinant of $\mathbf{A}$ by the second row:

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{2+1} a_{21} \det \mathbf{A}_{21} + (-1)^{2+2} a_{22} \det \mathbf{A}_{22} + (-1)^{2+3} a_{23} \det \mathbf{A}_{23}\text{, that is,}$$

$$\begin{aligned} \det \mathbf{A} &= -a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ &= -a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{22}(a_{11}a_{33} - a_{31}a_{13}) - a_{23}(a_{11}a_{32} - a_{31}a_{12}) \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}). \end{aligned}$$

Let us expand the determinant of $\mathbf{A}$ by the third column:

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+3} a_{13} \det \mathbf{A}_{13} + (-1)^{2+3} a_{23} \det \mathbf{A}_{23} + (-1)^{3+3} a_{33} \det \mathbf{A}_{33}\text{, that is,}$$

$$\begin{aligned} \det \mathbf{A} &= a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{13}(a_{21}a_{32} - a_{31}a_{22}) - a_{23}(a_{11}a_{32} - a_{31}a_{12}) + a_{33}(a_{11}a_{22} - a_{21}a_{12}) \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}). \end{aligned}$$

Indeed, the determinant of $\mathbf{A}$ may be expanded by any row or any column.

4.

(a) $3 \cdot 2 = 6.$ (b) $3 - 1 = 2.$ (c) $3 \cdot 2 \cdot (3-1) = 12$.

(d) $4 \cdot (3 \cdot 2) = 24$, $4 - 1 = 3.$, $4 \cdot 3 \cdot 2 \cdot (4-1) = 72.$

(e) The determinant of an $n \times n$ matrix is defined as the sum of n terms, where each term contain the determinant of an $(n-1) \times (n-1)$ matrix. Such a determinant is defined as the sum of $(n-1)$ terms, where each term contains the determinant of an $(n-2) \times (n-2)$ matrix and so on. The total number of terms in the sum: $n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 = n!$ The number of multiplications in each term is $(n-1)$. Hence to total number of multiplications that has to be performed is $n!(n-1)$.

(f) $n!(n-1) = 25!24 = 3.72 \times 10^{26}$.

(g) Total number of operations is the number of multiplications added with the number of additions, in total $n!(n-1) + n! - 1 = n!n - 1 = 3.88 \cdot 10^{26}$. The time required to evaluate the determinant $3.88 \cdot 10^{26} / 3.6 \cdot 10^{14} \text{s} \approx 1.08 \cdot 10^{12} \text{s} \approx 34{,}156 \text{ years}$.