Problems for Lecture 8: Linear Equations I & Determinants

1. Determine whether the following sets of linear equations have a unique solution and comment on your conclusion in each case:

(a) $\begin{aligned} 8x_1 + x_2 + 8x_3 &= 12 \\ 6x_1 + 4x_2 + 4x_3 &= 8 \\ 5x_1 - x_2 + 6x_3 &= 15 \end{aligned}$ (b) $\begin{aligned} 4x_1 + 7x_2 - 2x_3 &= 10 \\ x_1 - 3x_2 + 2x_3 &= 6 \end{aligned}$ (c) $\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 - 2x_2 + 2x_3 &= 0 \\ 4x_1 - 4x_2 - 4x_3 &= -1 \end{aligned}$

2. The two planes

$$x + 3y + z = 8$$
$$2x + y + 3z = 7$$

intersect in a line.

(a) Find the coordinates of the point where the line intersects the plane $y = 0$.

(b) Find a vector that is normal to each plane.

(c) Find a vector directed along the line of intersection.

(d) Use the results of (a) and (c) to obtain an equation for the line of intersection.

3. When the 3×3 determinant $\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is expanded after the first column, the result is $\det \mathbf{A} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13})$.

Expand the determinant after the second row and after the third column and show that the three results are identical.

4. Consider once again the determinant of the 3×3 matrix

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$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13})$$

(a) How many terms are there in the sum?
(b) How many multiplications have to be carried out to find each term?
(c) How many multiplications have to be carried out to calculate the entire determinant?
(d) Repeat (a)-(c) for a 4×4 determinant
(e) Repeat (a)-(c) for an $n \times n$ determinant.
(f) How many multiplications would be required to evaluate a 25×25 determinant?
(g) The fastest computer in the world can carry out about 360 Terra Flops = $3.6 \cdot 10^{14}$ operations per second. How long would it take it to evaluate a 25×25 determinant in this way?