Problems for Lecture 7: Answers

1. Let $\mathbf{r}$ be any point in the plane passing through $\mathbf{a}$. Then the vector $\mathbf{r}-\mathbf{a}$ is in the plane and hence normal to $\mathbf{n}$, implying $(\mathbf{r}-\mathbf{a})\bullet\mathbf{n}=0$ or $\mathbf{r}\bullet\mathbf{n}=\mathbf{a}\bullet\mathbf{n}$. We note that $\mathbf{r}\bullet\mathbf{n}=x+y-2z$ while $\mathbf{a}\bullet\mathbf{n}=1\cdot1+(-2)\cdot1+3\cdot(-2)=-7$ The equation of the plane is therefore $x+y-2z=-7$, or any multiple thereof.

2. (a) Since the points A,B, and C are in the plane, they must satisfy the equation $4x-3y+2z=7$. Substituting the coordinates of A, B and C in turn into the equation yields $4-3+2a=7 \Leftrightarrow a=3$, $8-3b+14=7 \Leftrightarrow b=5$, and $4c-15-10=7 \Leftrightarrow c=8$.

   (b) From the equation of the plane $4x-3y+2z=7$ we identify that the vector $\mathbf{n}=4\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ is normal to the plane. Its magnitude is $|\mathbf{n}|=\sqrt{4^2+(-3)^2+2^2}=\sqrt{29}$ so the unit normal vector is $\pm\hat{\mathbf{n}}=\pm\frac{\mathbf{n}}{|\mathbf{n}|}=\pm\frac{4\mathbf{i}-3\mathbf{j}+2\mathbf{k}}{\sqrt{29}}$. Either sign is acceptable.

   (c) $\overrightarrow{AC}=(8,5,-5)-(1,1,3)=7\mathbf{i}+4\mathbf{j}-8\mathbf{k}$ implying $\overrightarrow{AC}\bullet\mathbf{n}=7\cdot4+4\cdot(-3)+(-8)\cdot2=0$. Also, $\overrightarrow{BC}=(8,5,-5)-(2,5,7)=6\mathbf{i}-12\mathbf{k}$ implying $\overrightarrow{BC}\bullet\mathbf{n}=6\cdot4+0\cdot(-3)+(-12)\cdot2=0$. Hence we conclude that $\overrightarrow{AC}\perp\mathbf{n}$ and $\overrightarrow{BC}\perp\mathbf{n}$.

3. (a) $\begin{vmatrix}4 & 2\\ 1 & 5\end{vmatrix}=4\cdot5-1\cdot2=18$ and (b) $\begin{vmatrix}4 & 1\\ 2 & 5\end{vmatrix}=4\cdot5-2\cdot1=18$ too. The matrix association with case (b) is the transpose of that in case (a). Indeed, if $\mathbf{A}$ is an $n\times n$ matrix, then $\det\mathbf{A}=\det\mathbf{A}^t$ where the matrix $\mathbf{A}^t$ is the transpose of matrix $\mathbf{A}$, that is, the matrix obtained by reflecting the matrix $\mathbf{A}$ across its main diagonal. In other words, the ijth entry in $\mathbf{A}^t$ is equal to the jith entry in $\mathbf{A}$, that is, $a^t_{ij}=a_{ji}$. Hence, the value of a determinant is unchanged if rows and columns are interchanged.

   (c) $\begin{vmatrix}1 & 5\\ 4 & 2\end{vmatrix}=1\cdot2-4\cdot5=-18$. The two rows of the determinant in (a) have been reversed. Indeed, the sign of a determinant is reversed if two rows (or two columns) are interchanged.

   (d) $\begin{vmatrix}8 & 2\\ 2 & 5\end{vmatrix}=8\cdot5-2\cdot2=36$. Column 1 of determinant (a) has been multiplied by a factor of 2. Indeed, if the matrix $\mathbf{B}$ is obtained from the matrix $\mathbf{A}$ by multiplying some column (or row) by a number r, $\det\mathbf{B}=r\det\mathbf{A}$.

   (e) $\begin{vmatrix}4 & 2\\ 5 & 7\end{vmatrix}=4\cdot7-5\cdot2=18$. Row 1 of determinant (a) has been added to row 2. Indeed, if the matrix $\mathbf{B}$ is obtained from the matrix $\mathbf{A}$ by adding a numerical multiple of one row (column) to another, $\det\mathbf{B}=\det\mathbf{A}$.

4. $\begin{vmatrix} a & b \\ ca & cb \end{vmatrix} = a \cdot cb - ca \cdot b = 0$. Indeed, if any two rows (or colums) of a matrix $\mathbf{A}$ are proportional, $\det \mathbf{A} = 0$.

5. (a) $\begin{matrix} 3x+5y=14 \\ 2x+4y=10 \end{matrix} \Leftrightarrow \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 10 \end{pmatrix}$. The determinant of the matrix of coefficients is $\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 3 \cdot 4 - 2 \cdot 5 = 2 \neq 0$. Hence there is a unique solution and according to Cramer's rule $x = \frac{\begin{vmatrix} 14 & 5 \\ 10 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}} = \frac{14 \cdot 4 - 10 \cdot 5}{2} = \frac{6}{2} = 3$ and $y = \frac{\begin{vmatrix} 3 & 14 \\ 2 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}} = \frac{3 \cdot 10 - 2 \cdot 14}{2} = \frac{2}{2} = 1$. The two lines cross at the point $(x, y) = (3,1)$.

(b) $\begin{matrix} 3x-5y=8 \\ 7x+2y=12 \end{matrix} \Leftrightarrow \begin{pmatrix} 3 & -5 \\ 7 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$. The determinant of the matrix of the coefficients to the system of linear equations is $\begin{vmatrix} 3 & -5 \\ 7 & 2 \end{vmatrix} = 3 \cdot 2 - 7 \cdot (-5) = 41 \neq 0$. There is a unique solution. Cramer's rule yields $x = \frac{\begin{vmatrix} 8 & -5 \\ 12 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 7 & 2 \end{vmatrix}} = \frac{8 \cdot 2 - 12 \cdot (-5)}{41} = \frac{76}{41}$ and $y = \frac{\begin{vmatrix} 3 & 8 \\ 7 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 7 & 2 \end{vmatrix}} = \frac{3 \cdot 12 - 7 \cdot 8}{41} = -\frac{20}{41}$. The two lines cross at the point $(x, y) = \left(\frac{76}{41}, -\frac{20}{41}\right)$.

(c) $\begin{matrix} 6x+3y=9 \\ 4x+2y=6 \end{matrix} \Leftrightarrow \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$ The determinant of the matrix of coefficients is $\begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix} = 6 \cdot 2 - 4 \cdot 3 = 0$. Hence there is no unique solution. We notice that the two equations are proportional since the second can be obtained from the first by multiplication with 2/3. Therefore, the equations represent the same line and we have infinitely many solutions, namely all the points on the line $2x + y = 3.$.

(d) The associated determinant $\begin{vmatrix} 1.4 & -1.2 \\ -2.1 & 1.8 \end{vmatrix} = 1.4 \cdot 1.8 - (-2.1) \cdot (-1.2) = 0$. Hence there is no unique solutions. The two lines are parallel but have no points in common and there are no solutions to the pair is equations.