Problems for Lecture 6: Answers

1. Since the line passes through the points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, the vector $\mathbf{r}_2 - \mathbf{r}_1 = (8-3)\mathbf{i} + (-5-4)\mathbf{j} = 5\mathbf{i} - 9\mathbf{j}$ specifies the direction for the line. Hence, a vector equation for this line is $\mathbf{r} = \mathbf{r}_1 + \lambda(\mathbf{r}_2 - \mathbf{r}_1), \lambda \in \mathbb{R}$ that is, $\mathbf{r} = (3\mathbf{i} + 4\mathbf{j}) + \lambda(5\mathbf{i} - 9\mathbf{j})$ or, if you prefer, $\mathbf{r} = (3+5\lambda)\mathbf{i} + (4-9\lambda)\mathbf{j}$. Call this line A for future reference.

   Written in terms of components, this leads to $x = 3+5\lambda$ and $y = 4-9\lambda$. Isolating λ in these two equations, we find $\lambda = \frac{x-3}{5} = \frac{y-4}{-9}$. Note that the denominators are the associated coordinates of the vector specifying the direction of the lines, that is, the direction ratios for line A.

   Note that the vector $\mathbf{r}_1$ on the right hand side of the original vector equation could be the position vector of <u>*any*</u> point on the line, and that the vector multiplied by λ could be <u>*any*</u> vector parallel to the line, that is, any multiple $\mu(\mathbf{r}_2 - \mathbf{r}_1), \mu \neq 0$ would serve as the direction.

2. The gradient is 3, so the simplest choice of a vector specifying the direction of the line is $\mathbf{d} = (1,3)$. The line passes through the point $\mathbf{r}_0 = (x_0, y_0) = (0,-2)$ so a possible vector equation for this line is $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d} = \lambda\mathbf{i} + (-2+3\lambda)\mathbf{j}, \lambda \in \mathbb{R}$. On coordinate form, this yields $x = \lambda$ and $y = -2+3\lambda$. Hence we find $y = -2+3x$. Call this line B for future reference.

3. (a) The direction rations of line A are $(5,-9)$ and those of line B are $(1,3)$.

   (b) The direction cosines are found by normalising the vector specifying the direction of the line. Hence, since $|\mathbf{r}_2 - \mathbf{r}_1| = \sqrt{5^2 + (-9)^2} = \sqrt{106}$, the direction cosines of line A are $(\frac{5}{\sqrt{106}}, \frac{-9}{\sqrt{106}})$. Similarly, since $|\mathbf{d}| = \sqrt{1^2 + 3^2} = \sqrt{10}$, the direction cosines of line B are $(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})$.

   (c) Note that if $\mathbf{a} = (a_x, a_y)$ is a vector in $\mathbb{R}^2$ then the vectors $\mathbf{a}_\perp = (-a_y, a_x)$ and $-\mathbf{a}_\perp = (a_y, -a_x)$ are perpendicular to $\mathbf{a}$ since $\mathbf{a} \bullet \mathbf{a}_\perp = a_x \cdot (-a_y) + a_y \cdot a_x = 0$ and $\mathbf{a} \bullet (-\mathbf{a}_\perp) = a_x \cdot a_y + a_y \cdot (-a_x) = 0$. This leads to the following unit normal vectors for lines A and B $\hat{\mathbf{n}}_A = \pm\frac{9}{\sqrt{106}}\mathbf{i} \pm \frac{5}{\sqrt{106}}\mathbf{j}$ and $\hat{\mathbf{n}}_B = \mp\frac{3}{\sqrt{10}}\mathbf{i} \pm \frac{1}{\sqrt{10}}\mathbf{j}$, respectively.

(d) & (e) The angle θ between the lines A and B is the same as the angle between the normals $\hat{\mathbf{n}}_A$ and $\hat{\mathbf{n}}_B$. Since, in general, $\cos\theta = \frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$, we find $\cos\theta = \hat{\mathbf{n}}_A \bullet \hat{\mathbf{n}}_B = \frac{-22}{\sqrt{1060}}$ yielding $\theta = 2.31\,\text{rad} = 132.5°$. However, since the angle between two lines by definition is the smaller of the two, the answer is $\theta = 180^\circ - 132.5^\circ = 47.5^\circ$.

(f) The perpendicular distance to line A is $p_A = |\hat{\mathbf{n}}_A.\mathbf{r}_A|$, where $\mathbf{r}_A$ is any position vector on line A and similarly, the perpendicular distance to line B from the origin is $p_B = |\hat{\mathbf{n}}_B.\mathbf{r}_B|$, where $\mathbf{r}_B$ is any position vector on line B. Hence, we find that $p_A = \frac{47}{\sqrt{106}}$ and that $p_B = \frac{2}{\sqrt{10}}$.

4. The vector $\mathbf{r}_4 - \mathbf{r}_3 = (5,-3,7) = 5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$ defines the direction for the line passing through the point $\mathbf{r}_3$. A possible vector equation for the line is therefore $\mathbf{r} = \mathbf{r}_3 + \lambda(\mathbf{r}_4 - \mathbf{r}_3)$ which may be written on the form $\mathbf{r} = (2+5\lambda)\mathbf{i} + (1-3\lambda)\mathbf{j} + (-3+7\lambda)\mathbf{k}$. On component form we have $x = 2+5\lambda, y = 1-3\lambda$ and $z = -3+7\lambda$. Isolating λ yields $\lambda = \frac{x-2}{5} = \frac{y-1}{-3} = \frac{z+3}{7}$.

5. The direction ratios of the two lines $(2,5,3)$ and $(4,10,6)$ are proportional, and the direction cosines are therefore the same. The second line clearly goes through (10, 22, 13) and direct substitution shows that the first line does too. Since therefore the lines have a point in common, and are in the same direction, they are identical.

6. According to Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Hence, by raising the complex exponential to the power of n we find that

$$\left(e^{i\theta}\right)^n = e^{in\theta} \Leftrightarrow \left(\cos\theta + i\sin\theta\right)^n = \cos n\theta + i\sin n\theta.$$

(i) $z_1^{-1} = \frac{1}{3-4i} = \frac{3+4i}{(3-4i)(3+4i)} = \frac{3+4i}{3^2+4^2} = \frac{3}{25} + \frac{4}{25}i.$ Hence the real and imaginary parts are $\mathrm{Re}(z_1^{-1}) = \frac{3}{25}$ and $\mathrm{Im}(z_1^{-1}) = \frac{4}{25}$, respectively.

(ii) $|z_1| = \sqrt{3^2+(-4)^2} = \sqrt{25} = 5.$ $|z_2| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2.$ $\left|\frac{z_2}{z_1}\right| = \frac{|z_2|}{|z_1|} = \frac{2}{5}.$

To find the principal value of the arguments $-\pi < \arg(z) \le \pi$ of the complex number, we display them in an Argand diagram in conjunction and solving $\arg(z) = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)$ making sure the angle is in the correct quadrant.

$\arg(z_1) = \arctan\left(\frac{-4}{3}\right) = -0.927\text{rad} = -53.1^\circ$.

$\arg(z_2) = \arctan\left(\frac{1}{-\sqrt{3}}\right) = -0.52\text{rad} + \pi = 2.62 \text{ rad} = 150^\circ$, see first Argand diagram.

$\arg\left(\frac{z_2}{z_1}\right) = \arg(z_2) - \arg(z_1) = (2.62 + 0.927 - 2\pi) \text{ rad} = -2.74 \text{ rad}$, see first Argand diagram.

$\frac{z_2}{z_1} = z_2 z_1^{-1} = (-\sqrt{3}+i)(\frac{3}{25} + \frac{4}{25}i) = -\left(\frac{3\sqrt{3}+4}{25}\right) + \left(\frac{3-4\sqrt{3}}{25}\right)i \approx -0.368 - 0.157i$

(iii) On complex exponential form $z_2 = |z_2| e^{i\arg(z_2)}$ so $z_2^7 = |z_2|^7 e^{i7\arg(z_2)}$, that is, $|z_2^7| = |z_2|^7 = 2^7 = 128$; $\arg(z_2^7) = 7\arg(z_2) = 18.34 \text{ rad} = 18.34 \text{ rad } -3\cdot 2\pi = -0.51 \text{ rad}$ where we subtract a multiple of 2π to find the principal value of the argument.

(iv) We write z_1 on exponential form $z_1 = |z_1| e^{i(\arg(z_1)+2\pi n)}, n \in \mathbb{Z}$ from which

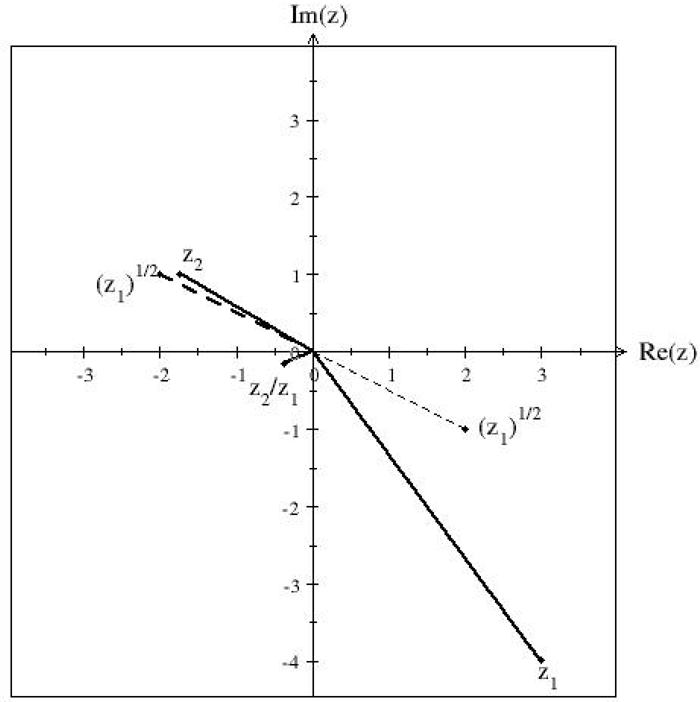
$z_1^{1/2} = \left(|z_1| e^{i\arg(z_1)+i2\pi n}\right)^{1/2} = |z_1|^{1/2} e^{i\arg(z_1)/2 + i\pi n} = \sqrt{5} e^{i(\arg(z_1)/2+\pi n)}.$

Hence, the two different values (solutions) are $\sqrt{5}e^{i\arg(z_1)/2} = \sqrt{5}e^{-i0.463} = 2-i$ and $\sqrt{5}e^{i(\arg(z_1)/2+\pi)} = \sqrt{5}e^{i2.68} = -2+i$. We easily check that $(2-i)(2-i) = 3-4i$ and likewise $(-2+i)(-2+i) = 3-4i.$

(v) See second Argand diagram

(vi) The equation $z^8 = 1$ has 8 different solutions according to the fundamental theorem of algebra.

(vii) $z^8 = 1 = e^{i2\pi n} \Leftrightarrow z = \left(e^{i2\pi n}\right)^{1/8} = e^{i2\pi n/8} = e^{i\pi n/4}.$ The different solutions are realised by letting $n = 0,1,2,3,4,5,6,7,$ that is, $z = 1, e^{i\pi/4}, e^{i\pi/2}, e^{i3\pi/4}, e^{i\pi}, e^{i5\pi/4}, e^{i3\pi/2}, e^{i7\pi/4}$.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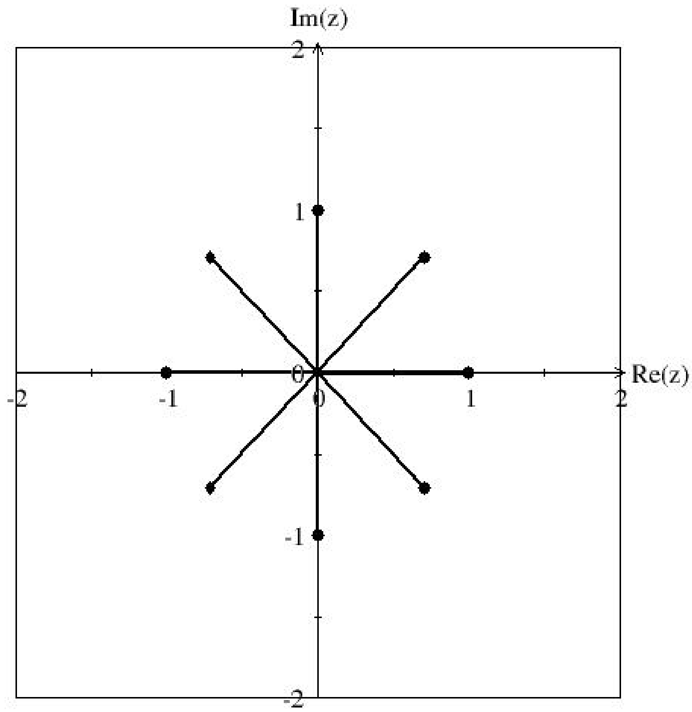
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