Problems for Lectures 17 & 18: Answers

1. (i) We note that $x(t)=Ae^{+\alpha t}+Be^{-\alpha t}\Rightarrow \dot{x}(t)=\alpha Ae^{+\alpha t}-\alpha Be^{-\alpha t}$. Hence $x(0)=A+B=x_0$ and $v(0)=\alpha A-\alpha B=v_0 \Leftrightarrow A-B=v_0/\alpha$. Solving these two equation w.r.t A and B yields $A=\dfrac{x_0+v_0/\alpha}{2}$ and $B=\dfrac{x_0-v_0/\alpha}{2}$. It follows that

$$\begin{aligned} x(t)&=\left(\frac{x_0+v_0/\alpha}{2}\right)e^{+\alpha t}+\left(\frac{x_0-v_0/\alpha}{2}\right)e^{-\alpha t}\\ &=x_0\left(\frac{e^{+\alpha t}+e^{-\alpha t}}{2}\right)+\frac{v_0}{2}\left(\frac{e^{+\alpha t}-e^{-\alpha t}}{2}\right)\\ &=x_0\cosh\alpha t+\frac{v_0}{\alpha}\sinh\alpha t. \end{aligned}$$

(ii) $B=\dfrac{x_0-v_0/\alpha}{2}=0\Leftrightarrow v_0=\alpha x_0$. Hence $x(t)=x_0\left(\cosh\alpha t+\sinh\alpha t\right)=x_0e^{+\alpha t}$.

If $x_0=1$ and $\alpha=1$ then $x(t)=e^t$.

2. We note that $x(t)=A\cos\left(\omega_0 t+\phi\right)\Rightarrow \dot{x}(t)=-\omega_0 A\sin\left(\omega_0 t+\phi\right)$. Hence $x(0)=A\cos\phi=x_0$ and $v(0)=-A\omega_0\sin\phi=v_0$. Taking the ratio, we find $\dfrac{v_0}{x_0}=-\omega_0\tan\phi$ implying $\tan\phi=-\dfrac{v_0}{\omega_0 x_0}=\dfrac{4}{3\cdot 2}=0.667$ which leads to $\cos\phi=0.832$ and $A=x_0/\cos\phi=2/0.832=2.404$. Inserting these values into the solution, we find $x(t)=2.404\cos\left(3t+\right)$.

3. (i) We note that $x(t)=Ae^{-\mu_- t}+Be^{-\mu_+ t}\Rightarrow v(t)=\dot{x}(t)=-\mu_- Ae^{-\mu_- t}-\mu_+ Be^{-\mu_+ t}$. Hence $x(0)=A+B=x_0$ and $v(0)=-\mu_- A-\mu_+ B=v_0$. Solving these two equations yields $A=\dfrac{\mu_+ x_0+v_0}{\mu_+-\mu_-}$ and $B=-\dfrac{\mu_- x_0+v_0}{\mu_+-\mu_-}$.

(ii)(a) In the case of strong damping, $\Gamma^2\gg\omega_0^2\Leftrightarrow\Gamma\gg\omega_0$, we find the following aproxiamtions $\mu_+=\Gamma+\sqrt{\Gamma^2-\omega_0^2}\approx\Gamma+\sqrt{\Gamma^2}\approx 2\Gamma$ and $\mu_-=\Gamma-\sqrt{\Gamma^2-\omega_0^2}=\Gamma\left(1-\sqrt{1-\omega_0^2/\Gamma^2}\right)\approx\Gamma\left(1-\left(1-\omega_0^2/2\Gamma^2\right)\right)=\omega_0^2/(2\Gamma)$ where we use the Taylor expansion $\sqrt{1-x}\approx 1-\frac{1}{2}x$ valid for $x\ll 1$.

(b) When $\Gamma\gg\omega_0\Rightarrow\mu_+\gg\mu_-$, and therefore $|A|\gg|B|$ and $x(t)\approx Ae^{-\mu_- t}\approx x_0e^{-\omega_0^2 t/2\Gamma}$

(c) If $\Gamma=30\ \mathrm{s}^{-1}$ and $\omega_0=2\ \mathrm{rad\ s}^{-1}$, the condition of strong damping applies. Writing $e^{-\omega_0^2 t/2\Gamma}=e^{-t/\tau}$ where τ is the characteristic time, it follows that $\tau=2\Gamma/\omega_0^2=15\,\mathrm{s}$. The function $x(t)\approx x_0e^{-\omega_0^2 t/2\Gamma}=x_0e^{-t/15}$.

4. Let $x(t) = Ae^{-\mu_- t} + Be^{-\mu_+ t}$. Inserting A and B, we find (see 3(i))

$$x(t) = \left(\frac{\mu_+ x_0 + v_0}{\mu_+ - \mu_-}\right) e^{-\mu_- t} - \left(\frac{\mu_- x_0 + v_0}{\mu_+ - \mu_-}\right) e^{-\mu_+ t}$$

$$= \frac{e^{-\mu_+ t}}{\mu_+ - \mu_-}\Big((\mu_+ x_0 + v_0) e^{(\mu_+ - \mu_-)t} - (\mu_- x_0 + v_0)\Big)$$

Now, as critical damping is approached, μ_+ and μ_- converge to Γ, and the difference $\mu_+ - \mu_- \to 0$, and therefore $e^{(\mu_+ - \mu_-)t} \approx 1 + (\mu_+ - \mu_-)t$. It follows that

$$x(t) = \frac{e^{-\mu_+ t}}{\mu_+ - \mu_-}\Big((\mu_+ x_0 + v_0) e^{(\mu_+ - \mu_-)t} - (\mu_- x_0 + v_0)\Big)$$

$$= \frac{e^{-\Gamma t}}{\mu_+ - \mu_-}\big((\mu_+ x_0 + v_0)(1 + (\mu_+ - \mu_-)t) - (\mu_- x_0 + v_0)\big)$$

$$= e^{-\Gamma t}\left(\frac{(\mu_+ - \mu_-)x_0 + (\mu_+ x_0 + v_0)(\mu_+ - \mu_-)t}{\mu_+ - \mu_-}\right)$$

$$= e^{-\Gamma t}\big(x_0 + (\Gamma x_0 + v_0)t\big).$$