Problems for Lectures 17 & 18: Second Order ODEs

1. The general solution to the second order ODE $\frac{d^2x}{dt^2} = \alpha^2 x$ is $x(t) = Ae^{+\alpha t} + Be^{-\alpha t}$.

   (i) Show that with the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$, then

   $x(t) = x_0 \cosh \alpha t + \frac{v_0}{\alpha} \sinh \alpha t$.

   (ii) What is v_0 when $B = 0$? Sketch the graph of $x(t)$ for $t \geq 0$, when $x_0 = 1$ and $\alpha = 1$.

   [*HINT*: See Fact Sheet 12 for definitions of cosh and sinh.]

2 The general solution to the second order ODE $\frac{d^2x}{dt^2} = -\omega_0^2 x$ is $x(t) = A\cos(\omega_0 t + \phi)$.

   If $\omega_0 = 3,\ x(0) = x_0 = 2,$ and $\dot{x}(0) = v_0 = -4$, find A and $\tan\phi$. Sketch the graph for $x(t)$.

3. We have studied the 2nd order ODE $\frac{d^2x}{dt^2} + 2\Gamma\frac{dx}{dt} + \omega_0^2 x = 0$. The motion is *overdamped* when $\Gamma^2 > \omega_0^2$, *critically damped* when $\Gamma^2 = \omega_0^2$, and *underdamped* when $\Gamma^2 < \omega_0^2$.

   In the case of *overdamped motion*, $\Gamma^2 > \omega_0^2$, the general solution was found to be $x(t) = Ae^{-\mu_- t} + Be^{-\mu_+ t}$, where $\mu_- = \Gamma - \sqrt{\Gamma^2 - \omega_0^2}$ and $\mu_+ = \Gamma + \sqrt{\Gamma^2 - \omega_0^2}$.

   (i) If $x(0) = x_0$ and $\dot{x}(0) = v_0$, show that $A = \frac{\mu_+ x_0 + v_0}{\mu_+ - \mu_-}$ and $B = -\left(\frac{\mu_- x_0 + v_0}{\mu_+ - \mu_-}\right)$.

   (ii) (a) In the case of *strong damping*, $\Gamma^2 \gg \omega_0^2$, show that $\mu_+ \approx 2\Gamma$ and $\mu_- \approx \omega_0^2 / 2\Gamma$.

   (b) Hence show that, if $v_0 = 0$ then $x(t) \approx x_0 \exp\left(-\omega_0^2 t / 2\Gamma\right)$.

   (c) If $\Gamma = 30\ \text{s}^{-1}$ and $\omega_0 = 2\ \text{rad s}^{-1}$, find the *characteristic decay time* (or 1/*e* time) of the system and sketch the function.

4. The solution of $\frac{d^2x}{dt^2} + 2\Gamma\frac{dx}{dt} + \omega_0^2 x = 0$ in the case of critical damping, $\Gamma^2 = \omega_0^2$, is of the form $x(t) = (A + Bt)e^{-\Gamma t}$, a result that is usually come out of the blue in many text books. In this question, you are invited to prove this result by starting from the solution to the overdamped case and then taking the limit $\Gamma^2 \to \omega_0^2$.

   By starting from the solution $x(t) = Ae^{-\mu_- t} + Be^{-\mu_+ t}$ in the case of overdamped motion, show that the solution for critical damped motion, $\Gamma^2 = \omega_0^2$, is given by $x(t) = \left[x_0 + (v_0 + \Gamma x_0)t\right]e^{-\Gamma t}$ where $x(0) = x_0$ and $\dot{x}(0) = v_0$.

   Note that the solution is, of course, consistent with $x(t) = (A + Bt)e^{-\Gamma t}$.

   [*HINT*: Substitute the expressions for *A* and *B* in question 3(i) into the solution for the overdamped motion, and take it towards the limit $\mu_+ \to \mu_- \to \Gamma$. To do this, factor out one of the exponentials, say $e^{-\mu_+ t}$, which will leave a term in $e^{(\mu_+ - \mu_-)t}$ that can be replaced by its Taylor (Maclaurin) expansion as the limit $\mu_+ \to \mu_- \to \Gamma$ is approached since $\left(\mu_+ - \mu_-\right)t \ll 1$. The result follows immediately.]