Problems for Lecture 16: Answers

1. To find the eigenvalues, we solve the characteristic equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$ and to find the associated eigenvectors, we solve the associated homogeneous equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ for each λ.

(i) $\det(\mathbf{A}-\lambda\mathbf{I})=0 \Leftrightarrow \begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix}=0 \Leftrightarrow (5-\lambda)(2-\lambda)-4=0 \Leftrightarrow \lambda^2-7\lambda+6=0$ so

$\lambda=\dfrac{7\pm\sqrt{7^2-4\cdot1\cdot6}}{2}=\dfrac{7\pm5}{2}=\begin{cases}6\\1\end{cases}$. We find the respective eigenvectors:

$\lambda_1=6$: $\begin{pmatrix}-1 & -2\\ -2 & -4\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Leftrightarrow\begin{cases}-x_1-2y_1=0\\ -2x_1-4y_1=0\end{cases}\Leftrightarrow x_1=-2y_1$ so an eigenvector associated with the eigenvalue $\lambda_1=6$ is $\mathbf{x}_1=\begin{pmatrix}-2\\1\end{pmatrix}$. Divide by $\sqrt{(-2)^2+1^2}=\sqrt{5}$ to normalise.

$\lambda_2=1$: $\begin{pmatrix}4 & -2\\ -2 & 1\end{pmatrix}\begin{pmatrix}x_2\\ y_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Leftrightarrow\begin{cases}4x_2-2y_2=0\\ -2x_2+y_2=0\end{cases}\Leftrightarrow y_2=2x_2$ so an eigenvector associated with the eigenvalue $\lambda_2=1$ is $\mathbf{x}_2=\begin{pmatrix}1\\2\end{pmatrix}$. Divide by $\sqrt{1^2+2^2}=\sqrt{5}$ to normalise.

The eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal, $\mathbf{x}_1\bullet\mathbf{x}_2=0$, because **A** is a real and symmetric matrix $\mathbf{A}^t=\mathbf{A}$.

(ii) $\det(\mathbf{B}-\lambda\mathbf{I})=0 \Leftrightarrow \begin{vmatrix} 5-\lambda & -7 \\ 1 & -3-\lambda \end{vmatrix}=0 \Leftrightarrow (5-\lambda)(-3-\lambda)+7=0 \Leftrightarrow \lambda^2-2\lambda-8=0$

so $\lambda=\dfrac{2\pm\sqrt{2^2-4\cdot1\cdot(-8)}}{2}=\dfrac{2\pm6}{2}=\begin{cases}4\\-2\end{cases}$. We find the respective eigenvectors:

$\lambda_1=4$: $\begin{pmatrix}1 & -7\\ 1 & -7\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Leftrightarrow\begin{cases}x_1-7y_1=0\\ x_1-7y_1=0\end{cases}\Leftrightarrow x_1=7y_1$ so an eigenvector associated with the eigenvalue $\lambda_1=4$ is $\mathbf{x}_1=\begin{pmatrix}7\\1\end{pmatrix}$. Divide by $\sqrt{7^2+1^2}=\sqrt{50}$ to normalise.

$\lambda_2=-2$: $\begin{pmatrix}7 & -7\\ 1 & -1\end{pmatrix}\begin{pmatrix}x_2\\ y_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Leftrightarrow\begin{cases}7x_2-7y_2=0\\ x_2-y_2=0\end{cases}\Leftrightarrow x_2=y_2$ so an eigenvector associated with the eigenvalue $\lambda_2=-2$ is $\mathbf{x}_2=\begin{pmatrix}1\\1\end{pmatrix}$. Divide by $\sqrt{1^2+1^2}=\sqrt{2}$ to normalise. Note that the eigenvectors are not orthogonal in this case.

(iii) $\hat{\mathbf{x}}_1=\mathbf{x}_1/|\mathbf{x}_1|=\begin{pmatrix}-2/\sqrt{5}\\ 1/\sqrt{5}\end{pmatrix}, \hat{\mathbf{x}}_2=\mathbf{x}_2/|\mathbf{x}_2|=\begin{pmatrix}1/\sqrt{5}\\ 2/\sqrt{5}\end{pmatrix}$.

(iv) The matrix of normalised eigenvectors: $\mathbf{S}=\left(\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2\right)=\begin{pmatrix}-2/\sqrt{5} & 1/\sqrt{5}\\ 1/\sqrt{5} & 2\sqrt{\sqrt{5}}\end{pmatrix}$.The transpose matrix $\mathbf{S}^t=\begin{pmatrix}-2/\sqrt{5} & 1/\sqrt{5}\\ 1/\sqrt{5} & 2/\sqrt{5}\end{pmatrix}$ and we note

$$\mathbf{S}^t\mathbf{S}=\begin{pmatrix}-2/\sqrt{5} & 1/\sqrt{5}\\ 1/\sqrt{5} & 2/\sqrt{5}\end{pmatrix}\begin{pmatrix}-2/\sqrt{5} & 1/\sqrt{5}\\ 1/\sqrt{5} & 2/\sqrt{5}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

(v) We know that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\Lambda=\mathbf{diag}(6,1)$. Hence, we find

$$\begin{aligned}\mathrm{Tr}\left(\mathbf{A}^{10}\right)&=\mathrm{Tr}\left(\mathbf{S}\mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{S}^{-1}\mathbf{A}\cdots\mathbf{S}\mathbf{S}^{-1}\mathbf{A}\right)\quad\text{since } \mathbf{S}\mathbf{S}^{-1}=\mathbf{I}\\
&=\mathrm{Tr}\left(\mathbf{S}(\mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{S}^{-1}\mathbf{A}\cdots\mathbf{S}\mathbf{S}^{-1}\mathbf{A})\right)\\
&=\mathrm{Tr}\left((\mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{S}^{-1}\mathbf{A}\cdots\mathbf{S}\mathbf{S}^{-1}\mathbf{A})\mathbf{S}\right)\quad\text{since Tr }\left(\mathbf{AB}\right)=\mathrm{Tr}\left(\mathbf{BA}\right)\\
&=\mathrm{Tr}\left((\mathbf{S}^{-1}\mathbf{A}\mathbf{S})(\mathbf{S}^{-1}\mathbf{A}\mathbf{S})\cdots(\mathbf{S}^{-1}\mathbf{A}\mathbf{S})\right)\\
&=\mathrm{Tr}\left(\Lambda\Lambda\cdots\Lambda\right)\quad\text{since } \mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\Lambda\\
&=\mathrm{Tr}\left(\Lambda^{10}\right)\\
&=6^{10}+1^{10}\end{aligned}$$

2. Let $\mathbf{x}$ be an eigenvector with eigenvalue λ for the matrix **B.** Then we find $\mathbf{B}^2\mathbf{x}=\mathbf{B}(\mathbf{B}\mathbf{x})=\mathbf{B}(\lambda\mathbf{x})=\lambda\left(\mathbf{B}\mathbf{x}\right)=\lambda\left(\lambda\mathbf{x}\right)=\lambda^2\mathbf{x}$. The eigenvector $\mathbf{x}$ for the matrix **B** is therefore also an eigenvector for the matrix $\mathbf{B}^2$ but with eigenvalue λ^2. Since the eigenvlues for **B** are $\lambda_1=4$ and $\lambda_2=-2$, the eigenvalues for $\mathbf{B}^2$ are $\lambda=\begin{cases}16\\ 4\end{cases}$.

3. (i) Consider a general diagonal matrix **A** with diagonal elements a_{11},a_{22} and a_{33}. The characteristic equations reads

$$\det\left(\mathbf{A}-\lambda\mathbf{I}\right)=0\Leftrightarrow\begin{vmatrix}a_{11}-\lambda & 0 & 0\\ 0 & a_{22}-\lambda & 0\\ 0 & 0 & a_{33}-\lambda\end{vmatrix}=0\Leftrightarrow\left(a_{11}-\lambda\right)\left(a_{22}-\lambda\right)\left(a_{33}-\lambda\right)=0 \text{ so}$$

the eigenvalues are the elements in the diagonal matrix $\lambda_1=a_{11},\lambda_2=a_{22,}$ and $\lambda_3=a_{33}$.

The eigenvector associated with $\lambda_1=a_{11}$ we find by solving the associated homogenous equation

$$\begin{pmatrix}0 & 0 & 0\\ 0 & a_{22}-\lambda & 0\\ 0 & 0 & a_{33}-\lambda\end{pmatrix}\begin{pmatrix}x_1\\ y_1\\ z_1\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Leftrightarrow\begin{cases}0=0\\ (a_{22}-\lambda)y_1=0\Leftrightarrow y_1=z_1=0\\ (a_{33}-\lambda)z_1=0\end{cases}\text{ so}$$

$\mathbf{x}_1=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_1=a_{11}$. Likewise we would find that $\mathbf{x}_2=\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$ and $\mathbf{x}_3=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$ are eigenvectors associated with $\lambda_2=a_{22}$ and $\lambda_3=a_{33}$, respectively. Hence, the eigenvalues and eigenvectors for the matrix **A** given in the question are the diagonal elements, that is, are

$\lambda_1 = 3$ associated with $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\lambda_2 = 5$ associated with $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\lambda_3 = 27$ associated with $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(ii) The characteristic equation

$$\det(\mathbf{B} - \lambda\mathbf{I}) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0 \Leftrightarrow -\lambda \begin{vmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 2-\lambda \\ 1 & 0 \end{vmatrix} = (\lambda^2 - 1)(2-\lambda) = 0$$

so the eigenvalues are $\lambda_1 = 2, \lambda_2 = 1$ and $\lambda_3 = -1$.

The eigenvector associated with $\lambda_1 = 2$ we find by solving the associated homogenous equation $\begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -2x_1 + z_1 = 0 \\ 0 = 0 \\ x_1 - 2z_1 = 0 \end{cases} \Leftrightarrow x_1 = z_1 = 0$ so $\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector.

The eigenvector associated with $\lambda_2 = 1$ we find by solving the associated homogenous equation $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -x_2 + z_2 = 0 \\ y_2 = 0 \\ x_2 - z_2 = 0 \end{cases} \Leftrightarrow x_2 = z_2, y_2 = 0$ so $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector.

The eigenvector associated with $\lambda_3 = -1$ we find by solving the associated homogenous equation $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x_3 + z_3 = 0 \\ y_3 = 0 \\ x_3 + z_3 = 0 \end{cases} \Leftrightarrow x_3 = -z_3, y_3 = 0$ so $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is an eigenvector.

(iii) The characteristic equation

$$\det(\mathbf{C} - \lambda\mathbf{I}) = 0 \Leftrightarrow \begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -1-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & 2-\lambda \\ 2 & 0 \end{vmatrix} = 0$$

Hence $(2-\lambda)(\lambda^2 - \lambda - 6) = 0$ so the eigenvalues are $\lambda_1 = 3, \lambda_2 = 2$ and $\lambda_3 = -2$.

The eigenvectors are $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ for $\lambda_1 = 3$, $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ for $\lambda_2 = 2$, and $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ for $\lambda_3 = -2$.

4. We find that $z_1 = 2 + 2i = \sqrt{8} e^{i(\pi/4 + 2\pi n)}, z_2 = -1 + 3i = \sqrt{10} e^{i(-1.893 + 2\pi n)}, n \in \mathbb{Z}$.

(i) $z_1^{10} = \left(\sqrt{8}\right)^{10} \left(e^{i(\pi/4+2\pi n)}\right)^{10} = 8^5 e^{i(5\pi/2+20\pi n)} = 8^5 e^{i5\pi/2} = 32768i$

(ii) $z_2^{-4} = \left(\sqrt{10}\right)^{-4} \left(e^{i(1.893+2\pi n)}\right)^{-4} = 10^{-2} e^{i(-7.57-8\pi n)} = 0.01 e^{-i1.287} = 0.0028 - 0.0096i$ Sign i?

(iii) $\left(z_1^*\right)^{10} = \left(z_1^{10}\right)^* = -32768i.$

5. Note that $i = e^{i(\pi/2+2\pi n)}$. Hence, $i^{1/7} = \left(e^{i(\pi/2+2\pi n)}\right)^{1/7} = e^{i(\pi/14+2\pi n/7)}, n \in \mathbb{Z}$. There are seven (7) different values corresponding to $n = 0,1,2,3,4,5,6$.

6. (i) Since $-1 = e^{i(\pi+2\pi n)}, n \in \mathbb{Z}$, we find that $\ln(-1) = \ln\left(e^{i(\pi+2\pi n)}\right) = i\pi + i2\pi n$, that is, $\ln(-1) = \ldots, -i5\pi, -i3\pi, -i\pi, i\pi, i3\pi, i5\pi, \ldots$. The principal value *is* $\mathrm{Ln}(-1) = i\pi$ *(n=0).*

(ii) Since $i = e^{i(\pi/2+2\pi n)}$, we find that $\ln(i) = i(\pi/2 + 2\pi n) = i\pi/2 + i2\pi n$, that is, $\ln(i) = \ldots, -i11\pi 2, -i7\pi 2, -i3\pi 2, i\pi/2, i5\pi 2, i9\pi/2, \ldots$. The principal value *is* $\mathrm{Ln}(i) = i\pi/2$ *(n=0).*

7. (i)

$$2^{i/2} = e^{\ln 2^{i/2}} = e^{\frac{i}{2}\ln 2} = e^{\frac{i}{2}(\mathrm{Ln}\, 2 + i2\pi n)} = e^{0.347i - \pi n} = e^{-\pi n} e^{0.347i}$$
$$= e^{-\pi n}(\cos 0.347 + i \sin 0.347) = e^{-\pi n}(0.941 + 0.340i).$$

(ii) Note that $1 + i = \sqrt{2} e^{i(\pi/4+2\pi n)}$. Hence we find that

$$(1+i)^{1+i} = \left(2^{\frac{1}{2}} e^{i(\pi/4+2\pi n)}\right)^{1+i} = 2^{\frac{1+i}{2}} e^{(i-1)(\pi/4+2\pi n)} = 2^{\frac{1}{2}} 2^{\frac{i}{2}} e^{i\pi/4} e^{-\pi/4 - 2\pi n}$$
$$= 2^{\frac{1}{2}} e^{-\pi n} e^{0.347i} e^{i\pi/4} e^{-\pi/4-2\pi n} = e^{-3\pi n}\left(2^{\frac{1}{2}} e^{-\pi/4}\right) e^{i(\pi/4+0.347)} = e^{-3\pi n}(0.274 + 0.584i)$$

8. (i) $\cosh x + \sinh x = \dfrac{e^x + e^{-x}}{2} + \dfrac{e^x - e^{-x}}{2} = e^x$.

(ii) $\cosh x - \sinh x = \dfrac{e^x + e^{-x}}{2} - \dfrac{e^x - e^{-x}}{2} = e^{-x}$.

(iii) $\cosh^2 x - \sinh^2 x = \left(\cosh x + \sinh x\right)\left(\cosh x - \sinh x\right) = e^x e^{-x} = e^0 = 1$.

(iv) $\sin iy = \dfrac{e^{i(iy)} - e^{-i(iy)}}{2i} = -i\dfrac{e^{-y} - e^{y}}{2} = i\dfrac{e^{y} - e^{-y}}{2} = i \sinh y$.

(v)

$$\sin(x+iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)}{2i}$$
$$= \sin x \frac{e^y + e^{-y}}{2} + i \cos x \frac{e^y - e^{-y}}{2} = \sin x \cosh y + i \cos x \sinh y.$$