

Problems for Lecture 16

Eigenvalue Problem and Complex Powers

1. Find the eigenvalues and eigenvectors of (i) $\mathbf{A} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$, (ii) $\mathbf{B} = \begin{pmatrix} 5 & -7 \\ 1 & -3 \end{pmatrix}$.

In one case, the eigenvectors are *orthogonal*. Identify this case and explain how this fact related to the structure of the matrix.

Considering this case only,

- (iii) Make the set of eigenvectors *orthonormal*.
 (iv) Create an orthogonal matrix and show explicitly that $\mathbf{S}^{-1} = \mathbf{S}^t$.
 (v) Calculate $\text{Tr}(\mathbf{A}^{10})$.
2. With $\mathbf{B} = \begin{pmatrix} 5 & -7 \\ 1 & -3 \end{pmatrix}$, deduce the eigenvalues of \mathbf{B}^2 .

3. Find the eigenvalues and eigenvectors of the transformations

$$(i) \mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 27 \end{pmatrix}, (ii) \mathbf{B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, (iii) \mathbf{C} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$

4. If $z_1 = 2 + 2i$ and $z_2 = -1 + 3i$, find (i) z_1^{10} , (ii) z_2^{-4} , (iii) $(z_1^*)^{10}$.

Express the answers in the form $x + iy$.

5. Find all values of $i^{1/7}$, that is, the 7th root of i .
 6. Find all values of (i) $\ln(-1)$, (ii) $\ln i$.

Identify the principal value in each case.

7. Find all values of (i) $2^{i/2}$ (ii) $(1+i)^{(1+i)}$ (Fiendish?! ☺).

Identify the principal value in each case.

8. Show that (i) $e^x = \cosh x + \sinh x$, (ii) $e^{-x} = \cosh x - \sinh x$,
 (iii) $\cosh^2 x - \sinh^2 x = 1$, (iv) $\sin(iy) = i \sinh y$,
 (v) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$.